BY THE SAME AUTHOR

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THE THEORY

OF

MEASUREMENTS

BY

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PREFACE

For the student of mathematics this book is intended to furnish an introduction to some of the applications of the exact sciences and their relation to the "practical" sciences and useful arts, and is primarily intended to give him a knowledge of facts and methods, but without neglecting the accurate exercise of his reasoning powers.

For the student of physical science it is intended especially to emphasize general considerations of measurement, theory of errors, general methods of procedure, quantitative accuracy, adjustment of observations, etc., —topics that are often merely mentioned in the introduction or appendix of a laboratory manual, but that need laboratory work and drill quite as much as the measurements of the individual quantities that the student will take up in his later work. Where it is impossible to find time for a complete course of the kind described here it may be helpful to use selected chapters of the book as occasion arises, or the student may be directed to use it as a reference book, or even to read it through without performing any of the experimental work.

The book is based on the mimeographed direction sheets that were used in the first part of a laboratory course which the writer gave at Jefferson at a time when there was no elementary textbook available that covered the required ground. In addition to the statements of facts and theory each of the chapters of the book includes directions for actual experimental work to be performed by the student, and the amount of this work has been so planned that each lesson will require

about the same length of time as any of the others. For the average student this will mean about three hours, but the material of the lessons can easily be divided into a greater number of shorter exercises if desirable.

With the more widespread use of a laboratory course of this sort certain shortcomings of the author's earlier "Introduction to Laboratory Physics" have been made manifest. A book that demands more or less vigorous mental exercise from a class of students who take a special interest in the subject will naturally need more elementary exposition—more detailed statement and less exercitational questioning—if it is to be used in larger classes where there is a greater likelihood of finding that some of the students are lacking in interest or ability or elementary mathematical training.* Accordingly, explanations and directions have been given with considerable detail, partly in order to avoid the necessity for continuous oral assistance on the part of the instructor, and partly to help the student to learn with a minimum of deliberate memorizing. For the latter purpose facts have occasionally been stated implicitly instead of explicitly, but, in such cases, always with a later reiteration in a more expositional form. The course is progressively graded in difficulty, with the object of developing the student's ability as he proceeds from the easier exercises to those that require more independent thought.

There has been a certain demand for the "Introduction to Laboratory Physics" in connection with courses of

* Such detailed directions as the instructions in regard to round numbers, page 119, may seem superfluous, but they indicate faults that have been found in the work of more than one of the students who have taken this course.

mathematics as well as for courses in physics, and for this reason the requirements of the mathematician have been especially kept in mind during the preparation of the present book. No knowledge of trigonometry, however, is presupposed, and none is imposed upon the reader of the book, the terms "function," "tangent," "cosine," etc., that will occasionally be found being used merely as convenient abbreviations for ideas that would otherwise need a more cumbersome description.

In the introductory chapter the commonest mathematical deficiencies of the student are reviewed and an opportunity is given him to test his weak points. A lesson on logarithms is included, which can be omitted, if preferred, by a class that is familiar with the subject; but there are often members of such a class who cannot make practical use of logarithmic tables readily, or even accurately, without additional practice, and to anyone who does not need the practice it will not be at all irksome. Care has been taken to make the tables in the appendix both accurate and convenient.* Experience has shown that the somewhat unconventional arrangement of the table of probable errors, page 292, is the most satisfactory in actual use. The table of logarithmic circular functions has been given the greatest possible compactness. The columns of the table of four-place logarithms are arranged especially for the convenience of students who are accustomed to using scales that are subdivided into tenths, and the proportional parts are given in the same way as in the most carefully constructed larger tables. None of the methods of arranging a five-place table with proportional parts within the limits of

* All of the tables either have been verified from two independent sources or have been checked by recalculation, and the proof-sheets have been revised with the utmost care.

two pages has ever succeeded in giving the fifth figure satisfactorily, and several scientific reference books have been published in which even the fourth figure of such a table will often be found incorrect. Accordingly, for the five-place logarithms in the present volume no attempt has been made to include proportional parts, but directions have been given for easy interpolation with the aid of three-figure logarithms.

I have replaced the perpetually misleading name for the common representative value of a set of residuals by one which does not have this objectionable quality and at the same time suggests the nature of the quantity in question. A few other innovations will be found here and there in the text, but for the most part the book follows fairly well-beaten lines.

I have found it advisable to devote the first ten or fifteen minutes of the laboratory period to a rapid recitation based on the lesson of the previous day; and have allowed the students to compare many of their important numerical determinations by having them record certain specified results each day upon a large card ($22\frac{1}{2}'' \times 28\frac{1}{2}''$) that is kept on one of the laboratory tables, and is ruled in separate columns headed by each student's name and having separate lines for each datum. For the exercises after the first chapter of this book the following data may be suggested:

Weights and measures: density of a (brass-and-air) weight.

Angles: largest error of the measured sines.

Significant figures: experimental value of π.

Logarithms: calculated value of e.

Small magnitudes: results of a double weighing.

Slide rule: approximate ratio for π, different from 22/7.

Graphic representation: temperature at 4 p. m.

Curves and equations: least value of x for which $\exp(-x^2)$ is indistinguishable from zero.

Graphic analysis: equation of the black thread experiment.

Interpolation: population, according to second extrapolation.

Coordinates in three dimensions: altitude at Sixth and Market Streets.

Accuracy: measured length of the table, or its relative deviation from the average.

Principle of coincidence: measured length of an inch.

Measurements: mode and extremes of measured variates.

Statistics: average, median, and quartiles of variates.

Dispersion: comparison of semi-interquartile range with dispersion (10 seeds).

Weights: weighted average for the density of aluminum.

Criteria of rejection: closer values for the approximate ratios 10 : 12 : 15.

Least squares: comparison of black thread determination (§ 21) with least square determination.

Indirect measurements: value of $\sqrt{3^2 + 5^2 + 6^2}$ by geometrical construction.

Systematic errors: direction and amount of displacement of the second hand.

Most of the apparatus required will be found to be included in that which is used in other physical experiments; a complete list of what is needed for each group of two students is given here:

2 metre sticks (graduated in tenths of an inch on the back).
2 30-cm. rulers (graduated in cm. and mm.).
1 50-cm^3 graduated cylinder.
1 10-cm^3 graduated pipette.

1 platform balance or trip scale with slide giving tenths of a gram (and the supporting wedges used when packed for shipment).
1 set of brass weights (1 gm. to 500 gm.).
1 set of iron weights (1 oz. to 8 oz.).
1 pair of fine-pointed dividers.
1 pencil-compass.
1 protractor (provided with a diagonal scale).
1 brass measuring disc for the determination of π.
2 10-inch slide rules that need not have celluloid facings, but are provided with A, B, C, D, S, L, and T scales, metric equivalents, and a runner.
1 hard wood block.
2 vernier calipers.
100 seeds or other variates.
1 aluminum block for density measurements.
1 set of "overflow can" and "catch-bucket" for Archimedes' principle.
2 square wooden rods for the balance pans.
1 iron clamp to hold balance on cross-bar over table.
1 irregular solid (large wire nail or strip of lead that can be immersed in the graduated cylinder).
1 small test-tube.
cardboard.
string.
fine black thread.

The student should have a watch with a second hand,* a pocket-knife, and the supplies mentioned in the introduction. A clock that beats audible seconds should be available. The slide rules should have 6745 on the C scale marked by making a shallow cut with a sharp knife and rubbing in a little oil pigment. The notebook (§ 8) used at the Jefferson Laboratory of Physics measures about eight by ten and a half inches and is ruled both horizontally and vertically at intervals of one seventh of an inch.

* A special watch for the laboratory, having a marked eccentricity of the second hand, may be advisable for the use of the students who have the greatest difficulty with the experiment on periodic errors.

CONTENTS

I. INTRODUCTORY

1. Object.—The object of a course in the theory of measurements is not only to give a certain knowledge of the scientific facts that are studied, but also to develop the thinking and reasoning powers and to furnish the special kind of mental training that results, in the first place, from practice in making various kinds of measurements with particular care for their accuracy, and, in the second place, from the consideration of accuracy in its quantitative aspects,—from realizing that accuracy itself can be made a subject of measurement, that there are relative degrees of accuracy, that accuracy is important in one place and means only a waste of effort in another, that absolute accuracy is an impossibility, that a measurement by itself is of much less value than when accompanied by a statement of its precision.

When a course of physical measurements is used as a preliminary to laboratory work or practical work in some subject such as astronomy, physics, psychology, or surveying it is further of value in giving the student a certain familiarity with apparatus and a facility in handling it in such a manner that he acquires the habit of utilizing it to the best advantage and of keeping it in such condition that it is most fully utilizable when needed.

2. Purpose.—It is advisable for the student to have it pointed out at the beginning of the course that his conscious purpose, throughout all of his work, should be to *learn*, rather than to accomplish the assigned exercise. In order to help him in his education and training he is permitted to do certain laboratory work which will make

his learning easier and its effects more lasting and more useful to him. He should tell himself that the work is allowed, rather than that it is required, in case he has any tendency to look upon his course of study as a tedious job, and thus make it irksome.

3. Continuity.—In any graded course of study, where each exercise is an advance beyond the previous ones and requires a knowledge of the earlier work, it is a decided handicap for the student to miss any of the work. If an absence cannot be avoided he should take particular pains, for his own sake, to make up the work by outside study. This is especially important for any study that is at all mathematical in character. See that each topic is thoroughly comprehended before going on to the next. The work will usually be easier if each lesson is read over before coming into class, so that it is not necessary to begin the class-room work as a new and unfamiliar subject.

4. Results.—In order to obtain good results it is necessary for the student to preserve an attitude of alert attention toward his own work, and especially not to omit any part of it or postpone it. Be thorough without being in haste; better to have half of the day's work done and done well than to try to take in all of it without having any of it more than half assimilated.

5. Forethought.—Whenever any piece of apparatus is used it must be kept in mind that it may be needed again later, and it is as important to keep it in good order as it is to use it efficiently. It does not take long for the consequences to show in the student's work if he is accustomed to pick up an instrument where he happens to find it, and drop it as soon as its immediate need has passed. A certain orderliness in the handling of apparatus is a habit that is well worth cultivating, as much

for its effect on the student's individuality as for the conservation of property values.

6. Mental Attitude.—In addition to keeping in mind the fact that it is much more important to learn principles than to "work through" each day's lesson the student should adopt the motto that in all kinds of studying it is better to *think* than to memorize. For some students it seems only too easy to get into the habit of concentrating upon individual items and memorizing isolated statements of fact without ever understanding their bearings or realizing their inter-relationships or acquiring a larger comprehension of the body of scientific knowledge which is built up on them. The best help to a broader vision lies in *thinking over* the facts that one comes across. Just as important as the question "What is true?" is the further question "Why is it true?" Better than a brain packed full of facts is a mind that can reason out what the facts must necessarily be in particular cases. Memory with little reasoning power is useless for any highly organized living being; but reasoning power with little memory would be perfectly practicable, as long as such things as paper and pencil can be had. Furthermore, the ability to think for oneself is of the greatest utility in enabling the student to rely upon his own observational powers. The untrained student is prone to ask "Is my result right?" in circumstances where the student who has learned to stand on his own feet knows that no one has a better knowledge of what the "right" result is than himself.

7. Notes.—For any practical work, or laboratory work, it is important that the student's notes shall be *accurate* and that they shall be *complete.* Neatness is usually worth while, but it is distinctly secondary in importance to thoroughness and accuracy. Time spent on beauti-

fying the notebook by means of elaborately shaded drawings or painstaking arrangement of matter is usually time wasted. All notes should be clear and intelligible, and in such shape that they can be readily understood by anyone who has an ordinary knowledge of the subject that they deal with. Each day's work is always to be dated, and it is advisable to write a heading in such a way that it will catch the eye at once, and show at a glance where one day's work ends and the next begins, as well as indicating the nature of the following matter after the manner of a title. A certain amount of "display," by underlining or otherwise should also be given to two other things, the statement of *original measurements*, especially when further calculations depend upon them, and the *final results* of such calculations.

The matter of making one's notes *thorough* needs little explanation; they should be made a digest of all that the student learns and does in the laboratory course, but without copying or duplicating matter in his textbook that he can easily turn to when it is wanted. Any questions that are asked in the text should be answered in the notebook.

The matter of accuracy is one that requires some care and alertness. It is necessary to make it a rule that all work done with a pen or pencil *must* be done in the notebook and everything in the notebook must be put down consecutively, in its natural order. If the student uses the last pages of the notebook for miscellaneous calculations it is usually impossible to find a particular piece of work when it happens to be wanted at some later time. Under no circumstances is any work with pen or pencil to be done on scraps of paper, or in a "temporary notebook" or anywhere else except in its proper place and order in the permanent notebook. The slight gain

in neatness of the student's notes, which is usually the object of such procedures, is not nearly important enough to counterbalance the possibility of errors in copying data and the probability of being later obliged to hunt in vain for statements that are not in their proper place. The notebook of the scientist, like that of the accountant, should be a book of "original entry," and for reasons that are as important for a scientific investigation of natural phenomena as they are for a legal investigation of indebtedness. Furthermore, no measurement that has been written down in the notebook should ever be rubbed out with an eraser; if there is a reason for doubting its value, or even if it is obviously wrong it may be canceled by drawing a line through it, but this should be done in such a way as not to obscure what is written down but to permit it to be utilized later if it is found desirable to do so.

8. Material Equipment.—The student's *notebook* should be of such size and character as will be best adapted to his work. If it is not furnished by the Department directions will be given in regard to the kind of notebook that should be used. *Pen and ink* will be needed, for notes that are taken with a pencil are almost always unsatisfactory. A fountain pen is advisable, although not necessary. A piece of *blotting paper* should be obtained which is long enough to reach across the page of the notebook. A *hard pencil* with the point kept well sharpened will also be needed.

9. Mental Equipment.—The student of the theory of measurement should have had a good course in algebra as far as the solution of equations of the first degree; also a sufficient knowledge of plane geometry to include the properties of perpendiculars, equal triangles, isosceles and similar triangles, the theorem of Pythagoras, and

the properties of similar figures. Certain arithmetical processes, such as the method of extracting square root or cube root, are not needed for purposes of physical measurement, but a good grasp of certain others, such as proportion and variation, is almost a necessity. An intelligent comprehension of principles is as important as a memory of rules and formulæ.

10. Proportionality.—The rule that the product of the means is equal to the product of the extremes is one that can be used for the solution of almost any problem in proportionality, but the important thing for the student is to understand the meaning of the combination of terms that constitutes what is called a "proportion." For example, the value of a commodity is proportional to its amount; thus, it might be that 13 lbs. : 26 lbs. :: 43c. : · · · c. The student who can handle such an example only by multiplying 26 by 43 and then dividing by 13 is wasting much of his time on mathematical drudgery that might be much more profitably devoted to the study of other subjects. *A proportion is by definition an equality of ratios;* the ratio of 13 lbs. to 26 lbs. is stated to equal the ratio of 43c. to some other number of cents, and a glance will show that the second weight is twice as large as the first, whence the second cost is twice 43c., namely 86c. A ratio is the same as a fraction or a quotient, and 13 lbs. : 26 lbs. means 13 lbs./26 lbs. or 13 lbs. ÷ 26 lbs., or 1/2. It is equally correct to state that the ratio of quantity to cost of a commodity is constant (*cæteris paribus*), so that the above proportion may just as well be written in the form 13 lbs. : 43c. :: 26 lbs. : 86c. Here the first ratio is not an abstract number like 1/2, but is a certain number (about .302) of *pounds for a cent;* and the second ratio, which is stated to be equal to it (:: means the same as =), must also be a number of

pounds for a cent. Some students may find it advisable to write every proportion that they use in the fractional form

$$\frac{13 \text{ lbs.}}{43\text{c.}} = \frac{26 \text{ lbs.}}{x} \quad \text{or} \quad \frac{13 \text{ lbs.}}{26 \text{ lbs.}} = \frac{43\text{c}}{x}.$$

In either of these the possibility of dividing by 13 is seen at once, and in this form no uncertainty can be felt in regard to what should be done with proportions like

$$13 : 43 :: 26 : x :: 39 : y :: 6.5 : z.\text{*}$$

Furthermore, there can be no difficulty in answering questions such as the following, which sometimes make trouble for the student who fails to realize the meaning of proportionality: "The pressure of a gas is stated to be proportional to its temperature. How can these *two* things be proportional if it always takes four terms to make a proportion?"

The *equality* of two or more ratios, which is the essential of proportionality, is often expressed by the use of some symbol to denote a constant or invariable quantity. Thus 13 lbs. : 43c. = .302 lb./c.; 26 lbs. : 84c. = .302 lb./c.; 6.5 lbs. : 21.5c. = .302 lb./c., and in general for this particular commodity *any weight* ÷ *corresponding value* = .302 lb./c. Following the usual custom of using the letter c or k to denote a constant it might be said of commodities in general that $w/v = k$. From this it follows that v/w is equal to $1/k$, which is another constant,

* In case any uncertainty *is* felt the student should attack it at once, and should not be satisfied until the difficulty has been successfully overcome. It is perhaps hardly necessary to point out the fact that a mathematical subject cannot usually be read as fluently as a novel. To have each letter and symbol observed by the eye, or even read aloud, is not enough unless the mind is given time for a thorough comprehension of the meaning.

say c, that is usually called "price." Since $1/.302 = 3.31$ the price of the substance considered above must be 3.31 cents per pound, which is a constant for that commodity, but of course the constant c (cost divided by weight) will have a value different from 3.31c./lb. if some other substance is considered. In technical language such a "variable constant" is called a *parameter*.

Is there any difference between what is stated by

$$\begin{cases} x_1 = cy_1, \\ x_2 = cy_2, \end{cases}$$

and what is stated by $x_1 : y_1 :: x_2 : y_2$? Prove it.

If the absolute temperature and the pressure of a given portion of gas are proportional what will happen to its pressure if the gas has its absolute temperature doubled or tripled?

11. Variation.—If the price of a commodity remains constantly 3.31 c./lb. the value is said to vary in accordance with the weight, or, shortly, to vary *as* the weight, or, more explicitly, to vary *directly as* the weight. Here the weight is considered to be a *variable quantity*, that is, we may consider any weight we please, the weight of the substance may assume any numerical value for the purposes of the discussion. Under these circumstances the cost will also vary. Doubling the weight will double the cost; cutting the weight in half will reduce the cost by 50 per cent; etc. When any change in one quantity that can vary is always accompanied by an equal relative change in a second quantity the variables are said to be proportional, or to be *directly proportional*, and each is said to vary *directly as* the other one.

If a portion of a gas is subjected to compression it will be found that doubling the pressure exerted upon it will cause its volume to decrease to only one half of its former

amount: multiplying the pressure by five will reduce the volume to one fifth; etc. This kind of variation is called *inverse,* and the pressure and volume are said to be *inversely proportional;* the volume is said to vary *inversely as* the pressure. Suppose that the volume is 6 quarts when the pressure is one atmosphere; then if the pressure is raised to 3 atmospheres the volume will be reduced to 2 quarts, but if it is diminished to 1/2 atmosphere the gas will expand enough to occupy a space of 12 quarts. If we write an equation

$$v = k\frac{1}{p}$$

it will be evident that any increase in the size of the denominator, p, will cause a relatively equal decrease in the size of the term, v; and we have already seen that this equation means the same as

$$v_1 : 1/p_1 :: v_2 : 1/p_2 :: v_3 : 1/p_3 \cdots.$$

Clearing the equation of fractions gives

$$pv = k$$

and the ordinary way of expressing the fact that two variables, such as p and v, are inversely proportional is to write down an equation in which their product is stated to be equal to a constant; just as direct proportionality is expressed by making their quotient equal to a constant.

It is perhaps worth noticing here that there may easily be other forms of variation, in which there is no proportionality at all. The distance traveled by a train is not usually strictly proportional to the number of hours that elapse during the process, nor is an individual's wealth proportional to his age. The matter of irregular variation will be taken up later.

12. Algebraical Formulæ.—The following are some of the facts of algebra which experience has shown that students of the theory of measurement need but are not in every case familiar with:

Letters are used for generalized numbers; if $(a+b)(a-b) = a^2-b^2$ then $(20+1)\times(20-1)=20^2-1^2$, and similarly for any other numbers.—"Terms," between plus or minus signs, are to be evaluated before performing the additions or subtractions; thus $2+4\times3-1+4(3-1)$ is equal to 21, not to 31 or any other number.—The product $a^3 \times a^4$ is a^7, not a^{12}; this is obvious if it is written or considered as $(a \times a \times a) \times (a \times a \times a \times a)$.—The product of a negative and a positive number is negative, but of two negatives is positive; *e. g.*, $(a + b - c)(x - y) = ax - ay + bx - by - cx + cy$.—A negative exponent indicates a reciprocal; a^{-2} means $1/a^2$.—A fractional exponent indicates a root; $a^{\frac{1}{2}}$ means $\sqrt{a}$; $x^{\frac{3}{4}} = \sqrt[4]{x^3} = (\sqrt[4]{x})^3$.—Fractional radicands may be simplified as shown in the following example:

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{a}}{\sqrt{b}}\,\frac{\sqrt{b}}{\sqrt{b}} = \frac{\sqrt{ab}}{\sqrt{b^2}} = \frac{1}{b}\sqrt{ab}.$$

—The binomial theorem is

$$(a + x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1 \cdot 2}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-3}x^3 + \cdots;$$

e. g.,

$$(a + b)^2 = a^2 + 2ab + b^2;$$
$$(m - n)^3 = m^3 - 3m^2n + 3mn^2 - n^3; \text{ etc.}$$

13. Mental Exercises.—The following list of exercises covers various subjects that students have occasionally been found lacking in familiarity with, also many that

will be needed in different parts of the course. The student should read each question and decide upon the answer mentally and without hesitation. If the answer is instantly apparent mark the question with a check ($\surd$) and take up the next one, but if it occasions any hesitation or uncertainty mark it with a plus sign (+) and if it cannot be answered at all inside of a few seconds mark it with a zero (0). Go through the whole list rapidly, and then ask the advice of the instructor in regard to it. This will often make a great difference in the ease of performing the later laboratory work. Remember that it is not an examination of how much it is possible to recall from the depths of your memory, but a test of how much mathematics you have in an immediately available condition.

1. What is the square of $x - a$?
2. What is the value of $(m + x)(m - x)$?
3. State the value of 2^3.
4. What is the numerical value of π?
5. Which is the larger 37/147 or 38/148? Do you know of any general method of deciding such a question?

Write the following in the form of decimal fractions:

6. 1/3.
7. 4/5.
8. 1/7.
9. 1/8.
10. 2/9.
11. 1/11.
12. Can the constant π be called a parameter? Why?
13. Reduce 1/25 to hundredths mentally.
14. Is 6 twice as large as 4? How many times as large?
15. What is the fourth term of 1000 : 100 :: 31 : . . .?
16. State the value of 1/500 as a decimal fraction.
17. Simplify the following: $(a^7)^5$; $a^7 + a^5$; $a^7 \times a^5$.

18. State the cube of $a + b$.

19. $(100 - 12) \times (100 + 12) = \cdots$?

20. If the circumference of a circle is 15 cm. in what way can the diameter be expressed?

21. Twenty inches on a certain map represents 2,000 miles. What is its scale of miles per inch numerically equal to?

22. State "one out of every four" as a percentage.

23. What does "twenty percent" mean?

24. Reduce 0.375 to a percentage.

25. What is the reciprocal of 2/7?

26. What percent is the number .005 equal to?

27. What percent is .00072?

28. $2.84 \times 10^{-4} = \cdots$?

29. $284 \times 10^{-4} = \cdots$?

30. Find $(1 + 1)^4$ by the binomial theorem.

31. Solve $3 : 12 :: 16 : x$.

32. How much is $(-2)(-15)/(-5)$?

33. What is the value of $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$?

34. Solve mentally $4^2 : x^2 :: 25^2 : 75^2$.

35. Is

$$\frac{a - b}{a} = 1 - \frac{b}{a}$$

true for numerical values? Give an example of it.

36. If pv is a constant how will v be affected by doubling p?

37. $3 + (4 - 4 \div 2)(7 \times 4 - 3) = ?$

38. $1 + 1 \cdot 2 + 1 \cdot 2 \cdot 3 + 1 \cdot 2 \cdot 3 \cdot 4 = ?$

39. State the approximate value of $8.5 \div 10$.

40. Does $5^2 \times 9^2$ equal 45^2? Use algebraical letters to illustrate the general principle that is involved.

41. State the *approximate* square root of each of the following: 2560 (ans.: about 50); 256; 25.6; 2.56; 0.256.

42. What is the approximate value of .01/2.38?

43. Write "49.78 thousandths of a centimetre" in the form n cm., where n is a decimal fraction.

44. Does

$$\frac{13.29 \times 0.81}{3826/41} \times 477$$

have a value of about 5, or about 50, or about 500?

45. In the equation

$$\frac{x}{a} + \frac{y}{b} = 1$$

what is the value of y if x is zero?

46. In the equation $y = 2x + 4$ what is the value of y when $x = -3$?

47. Complete the following sentence in the obvious way: A cubic inch of lead weighs 165 gm., and is 150 gm. heavier than 1 cubic inch of water; therefore . . . .

48. Substitute $1/m$ mentally for $-\log y$ in the expression $1/(-\log y)$ and simplify.

49. Write the value of $.0011 \times .00011$.

50. Solve $p = 2\pi\sqrt{l/g}$ for l, and then for g.

51. If

$$n = \frac{1}{2l}\sqrt{\frac{t}{sr}}$$

how does n change if l becomes $\frac{1}{4}$ of its former size? If t becomes $\frac{1}{4}$ as large as before? If s is $\frac{1}{4}$ as large?

52. Fill out the following table:

n	n^2	$2^n - 3$	$1/n$
0			
1			
2			
3			
$2x$			

14. Physical Arithmetic.—Before performing any written calculation with numbers that have been obtained

from physical measurements it is advisable to make a rough *mental* calculation of the approximate value of the final result. For example, at 3.31c. per lb. what will $5\frac{1}{2}$ lbs. cost? One of these factors is a little more than three; the other is somewhat less than six. Their product, accordingly, must be in the neighborhood of 18. A train travels 155.8 miles in $2\frac{3}{4}$ hours; what is its rate in miles per hour? Here the time is less than 3 hours, so the speed must be greater than 155.8 ÷ 3; and still greater than 150 ÷ 3. Ans.: Somewhat faster than 50 miles per hour. If a still closer approximation should be desired it could be obtained by noticing that the actual distance is about 4 percent greater than 150, and the actual time is one twelfth (say 6 percent) less than the assumed time. Increasing 50 by 4 percent and then by 6 percent would make it about 10 percent larger, giving 55 miles per hour for a closer value of the speed. The arithmetically accurate value is 56.65454545.

Is

$$\frac{4.71 \times 13.8}{9.06}$$

equal to about 7, or about 70, or about 700?

What is the approximate value of $7.26 \times .0328$?

Point off 00000928800000 so as to make it equal to the product of .0216 and .0043

Point off the right-hand side of the equation 2./.3 = 00666.

Reduce the ratios in the following expressions to approximate percentages, performing the calculation mentally: "8 lbs. in every 23 lbs. of sea water is solid salt" (ans.: 8 in 24 would be $33\frac{1}{3}$ percent; 8/23 is a little greater and must be 34 or 35 percent; or: $8/23 = 16/46$, $16/46 > 15/45$, $\therefore 8/23 > 1/3$); "seven inhabitants out of every

38 are voters" (ans.: 7 out of 35 would be 20 percent, 7/42 = $16\frac{2}{3}$ percent, 7/38 must have some intermediate value, say 18 percent); "fourteen-carat gold is 14/24 pure" (ans.: 14/24 = 28/48 = 56/96; this fraction has its numerator about half as large as its denominator and so will not be much changed by adding 4 to the latter if 2 is added to the former, 56/96 = 58/100 = 58 percent); "boiling water will dissolve .000022 of its weight of silver chloride" (ans.: .000022 = .0022 percent); "a saturated salt solution has a strength of 5 : 13" (ans.: a little less than 5 : $12\frac{1}{2}$ or 10 : 25 or 40 : 100, say 38 percent); "a steep railroad grade may have a rise of as much as 180 feet per mile" (ans.: 180 ft. per 5280 ft. is less than 180 per 5,000 or 360 per 10,000 or 36/1000 or .036 or 3.6 percent).

Notice the different expressions that are in common use to denote the comparison between a definite fractional part and the total. The same meaning is expressed by each of the following phrases as by any of the other ones: 22 per million, 22 out of a million, 22 in 1000000, 22/1000000, .000022, 22 : 1000000, and .0022 *per centum* or .0022 percent.

Observe also that the fractional notation is more convenient than the word *per* in naming compound units of measurement, and has the same significance. Thus, a speed of 40 miles per hour is customarily written $40 \frac{\text{mi}}{\text{hr}}$, or 40 mi/hr, meaning 40 times 1 mile per hour or $40 \frac{1 \text{ mile}}{1 \text{ hour}}$. That this notation is consistent will be made obvious by considering that 40 × 1 mile × 1 hour would necessarily be the same as 40 × 1 mile × 60 minutes, which reduces to 2400 × 1 mile × 1 minute, and 40 miles per hour is quite different from 2400 miles per minute; but 40 × 1

mile ÷ 1 hour = 40 × 1 mile ÷ 60 minutes = $\frac{2}{3}$ × 1 mile ÷ 1 minute, and $\frac{2}{3}$ of a mile per minute is plainly the same speed as 40 miles per hour. Of course there are some kinds of compound units which are properly expressed when their component simple units are multiplied together instead of being divided. For example, one foot-pound of energy is equal to 16 foot-ounces, a fact that could not be true if the unit were 1 × 1 foot ÷ 1 pound, but that requires it to be 1 × 1 foot × 1 pound. (Insulation resistance in "ohms per mile" and lighting efficiency in "watts per candle" furnish illustrations of *mis-named* units. The longer wire has the lesser insulation resistance so that 20 ohms for one mile is the same as 10 for two miles and the only rational name for the unit is the *ohm-mile.* Efficiency means light per energy and its unit would properly be *candle-power per watt,* but the illuminating engineer prefers to consider *inefficiency,* which is properly measured by watts per candle-power.)

15. Abridged Division.—A number that is obtained as the result of a physical measurement is frequently needed for some kind of a calculation. When this is the case it is a fact (as will be shown later) that the final result never needs to be expressed with a greater number of figures than the original data contained. Thus, it may be possible to measure the width of a table so carefully as to make sure that the measurement is 62 centimetres + 3 millimetres + 8 tenths of a millimetre. Such a quantity is preferably written as a number of centimetres, and in this case is 62.38, a number consisting of four figures. Suppose it is necessary to find out what one third of the width will amount to. One third of 62.38 is 20.793333..., and, as stated above, four figures of this result, namely 20.79, are all that are necessary. As a

matter of fact, to keep more than four figures would be decidedly objectionable. If the original measurement gave the correct number of tenths and hundredths of a centimetre without pretending to state any knowledge of the correct number of thousandths how could any calculation assume to give correct figures in thousandth's and tens-of-thousandth's places? Similarly, in § 14, the "arithmetically accurate" value would be wrong if the given distance were even a thousandth of an inch longer or if the time varied from an exact $2\frac{3}{4}$ hours by as much as a millionth of a second. Here one of the numbers (155.8 miles) has four figures while the other can hardly be considered to have more than three (2.75 hours). In such cases it is a fact that the final result will have only as many trustworthy figures as there are in the *shortest* number from which it is derived. When a number having four figures is divided by a number of three figures there should be only three figures kept in the quotient.

The principle just stated makes it possible to employ the "abridged" processes of multiplication and division, which will automatically give just the right number of figures in the answer, and will also save considerable labor on the part of the computer.

The first example shown in the margin has been worked out by ordinary "long" division; in the second one

```
275)1558(566545
    1375
     1830
     1650
      1800
      1650
       1500
       1375
        1250
        1100
         1500
         1375
          125
```

$2\cancel{7}\cancel{5})1558(567$

```
    1375
     183
     165
      18
      19
```

Abridged Division.— After each subtraction the next step is to shorten the divisor instead of to lengthen the dividend.

the abridged method has been used. The latter process differs from the former in only one respect: Whenever the process of "bringing down" a zero would be employed the last figure of the divisor is canceled instead. In order that the temporary dividend shall be larger than the divisor one method stretches out the dividend by affixing a cipher; the other shortens the divisor by trimming off its last figure. A comparison of the two examples will show that the same result is achieved in each case. The beginner should work out the quotient of the two numbers given above, canceling the last figure of the divisor whenever he would otherwise "bring down" a zero, but not referring to the illustration until he was finished: After the first subtraction, when the divisor, 275, is not contained in the remainder the divisor is shortened by crossing off the final 5. Then, 27 is contained in 183 six times. The next step is to multiply 27 by 6. Before saying $6 \times 7 = 42$ notice that if the 5 had not been canceled there would have been 3 to carry, resulting in 45 instead of 42. Accordingly 5 is written down instead of 2 and the rest of the multiplication proceeds as usual. After the next subtraction gives a remainder of 18 the divisor is shortened to 2 instead of having the new dividend lengthened to 180. Then, 2 would be contained in 18 just 9 times, but considering the figure that was last canceled it is plain that 2.7 will not be contained much more than 6 times. Six times the canceled 7 would be 42 and would give 4 to carry, so 6 times the 2 (plus 4) is written 16. The next subtraction and cancellation puts an end to the work. In the specimen given above it has been noticed that the third figure of the quotient is to be the last one, and it has been written down as a 7 instead of a 6 because the next product, 19, comes nearer in value to the re-

quired 18 than would the number 16. In other words the quotient is nearer 567 than 566 and so the larger number represents it more accurately than the smaller.

The quotient is to be pointed off by making a preliminary mental calculation, as explained in § 14. For example if the original numbers had been 2.75 and 155.8 the answer would have been 56.7. Similarly, 1558./.0275 = 56700; 1.558/27.5 = .0567; etc.

$$
\begin{array}{r}
3.\cancel{1}\cancel{4}\cancel{1}\cancel{6})180.000(572957 \\
\underline{157080} \\
22920 \\
\underline{21991} \\
929 \\
\underline{628} \\
301 \\
\underline{283^*} \\
18 \\
\underline{16} \\
2
\end{array}
$$

Abridged Division. — At the mark * notice that 3.6 comes nearer to being "4 to carry" than "3 to carry," since it is more than three and a half.

Divide 180.000 by 3.1416 without referring to the work given in the margin until the answer has been obtained. The sixth figure which is given in the quotient is intended to represent the value of the 2/3 remaining after the last subtraction. It could also have been obtained by continuing the regular process of abridged division: cancel the 3, leaving $0.\cancel{3}$ for the divisor; then $2 \div 0.3 = 7$; multiplying, $7 \times \cancel{3}$ gives 2 to carry, $7 \times 0 + 2 = 2$.

Find the quotient if the divisor is 236453 and the dividend is 6764309. The answer should come out 28.60741. After the figure 4 of the quotient has been written down the next step is to multiply $23\cancel{6}\cancel{4}\cancel{5}\cancel{3}$ by 4. Ordinarily it is sufficient to take the nearest canceled figure and say $4 \times 6 = 24$, giving 2 to carry; but as the product comes close to 25, which is on the boundary between 2 to carry and 3 to carry, it is well to investigate one more canceled figure, saying $4 \times \cancel{4} = 16$, giving 2 to carry toward $4 \times \cancel{6}$; the latter then becomes 26, giving 3 to

carry toward the written product instead of 2. Make up and work out two examples in which a long number is divided by a short number, and *vice versa*. Follow the regular routine: divide, multiply, subtract, and cancel; and repeat as many times as necessary.

16. Abridged Multiplication.—The trustworthiness of a product of two or more numbers follows the same rule as that of a quotient: no more figures of the product are "significant" than the number of them which the shortest factor contains. Thus if the diameter of a circle is found to be 8 centimetres + 0 millimetres + 0 tenths of a millimetre but nothing is stated about hundredths of a millimetre, so that only three figures, 8.00, of the diameter are known, it will not be possible to obtain more than three figures of the circumference, even if the other factor, 3.14159265359, contains a dozen figures. Here too the ordinary arithmetical process can be abridged so as to save time and work, and, what is more important, to avoid being misled by figures that have been kept when they should have been discarded.

The illustration (*a*) shows the ordinary process. It is just as easy, although not customary, to use the figures of

(*a*)	(*b*)	(*c*)
65.97	65.97	$6\cancel{5}.\cancel{9}\cancel{7}$
24.13	24.13	24.13
19791	13194	13194
6597	26388	2639
26388	6597	66
13194	19791	20
1591.8561	1591.8561	1591.9

GENESIS OF THE ABRIDGED METHOD.—(*a*) Ordinary long multiplication. (*b*) The same with the figures of the multiplier used in the reverse order. (*c*) The same as (*b*), but the partial products are kept from stringing out to the right by progressively shortening the multiplicand.

the multiplier in the reverse order, multiplying first by 2, then by 4, 1, and 3, and "stepping" the partial products successively one more place to the right instead of to the left. This has been done in illustration (*b*). Examine it closely, and see that you understand just how the process (*b*) is carried out and why it must necessarily give the same result as (*a*).

The abridged method is shown at (*c*). The first multiplication is by the left-hand figure 2 as in (*b*). Then the last figure of the multiplicand, 7, is canceled, and the next multiplication (by 4) is begun directly under the first. The process of canceling and multiplying is continued in the same way until either the multiplicand is entirely canceled or the multiplier has been entirely utilized, and the result is pointed off in accordance with the directions in § 14. The student should work out the same example independently, remembering to investigate how much there is "to carry" from the figure last canceled. In the last partial product of (*c*) the $3 \times 6 = 18$ is increased by two units because $3 \times \cancel{5}$ gives just 1.5 to carry but it is evident that $3 \times \cancel{5}\cancel{9}$ must give a number nearer to 2 than to 1.

Multiply the quotient 28.60741 given above by the divisor 236453, and notice that the dividend is found correctly as far as six figures (more than $676430\frac{1}{2}$), which is all that can be expected if one factor contains only six figures.

The value of $180/\pi$ is 57.29578. Multiply this by 3.141593 and see if you obtain 180.0000 correct to seven figures.

Multiply any number that has five figures by some number having only two figures. Repeat the multiplication, using the shorter number for multiplicand and the longer one for multiplier. Which method is pref-

erable, in view of the statement at the beginning of § 16?

17. Gradient.—If a road that goes up-hill rises 2 feet for every 5 feet of horizontal distance it is said to have a slope of 2 : 5, or 2/5, or 0.4, or 2 in 5. That is, the ratio of any vertical rise to the corresponding horizontal distance is taken as a numerical measure of its steepness. Of course the slope could also be measured in degrees; in the case just mentioned the "*grade angle*," or angle which the slant line makes with a horizontal line, would be twenty-two degrees—too steep to be satisfactory for a road-way,—but the usual custom is to state the measure of a slope in terms of vertical rise per horizontal distance. Another way of looking at the same thing is to consider it as the amount of rise per *unit* of horizontal distance; thus, a road that rises two feet in every five will of course have a rise of 2/5 of a foot for a single foot of horizontal distance.

A level road is one that has no slope. That is, its slope, when measured as rise per level distance, amounts to zero. If a line is made to slant more and more steeply the ratio that represents its slope will obviously become greater and greater without limit; thus, a slope of 1000 would be hardly distinguishable from a true vertical, and yet between these two there must be, for example, a slope of 1000000000. The diagonal of a square is inclined to any of the sides at an angle of 45°, and its slope is of course unity. These facts may be abbreviated as follows: slope of $0° = 0$; slope of $45° = 1$; slope of $90° = \infty$.

Draw roughly an equilateral triangle that has one side horizontal. Draw a vertical from its apex to the middle of its base, and prove that the gradient of a sixty-degree slope is equal to $\sqrt{3}$; also that if the angle is half as large as this the slope will be only one third as much.

It is sometimes convenient to consider the steepest

possible "slope" (a vertical line) as having 100 percent of steepness. This will be the case if we measure the amount of slant not by the ratio of rise to level distance but by the ratio of rise to slant distance. A road which has a grade angle of 22° will rise nearly 3 feet for every 8 feet of distance along its slanting surface, and this measure of steepness may be called *percent slope* to distinguish it from the slope, or gradient, as previously defined.

Prove that the "percent slope" of 45° is $\frac{1}{2}\sqrt{2}$; of 0° is zero; of 90° is 1; of 30° is $\frac{1}{2}$ (use the same bisected equilateral triangle).

The steepest slopes that are generally used for roadways range from 12 percent to 15 percent. On good turnpikes the grades are almost always kept below 3 percent. Two percent is decidedly steep for a railroad grade, and in modern good railroad construction one percent is about the maximum. For slight inclinations, such as railroad grades, the difference between rise per horizontal distance and rise per slant distance is negligibly small. Thus, for 1° they are respectively .017455 and .017452, or 92.16 feet per mile and 92.15 feet per mile.

II. WEIGHTS AND MEASURES

Apparatus.—Ruler; metre stick; graduated cylinder; graduated pipette; pair of dividers; irregular solid; platform balance; set of gram weights; set of ounce weights; towel; glass jar or "catch-bucket"; small test tube.

18. C.G.S. System.—The older units of measurement, such as the length of a barleycorn, the width of a man's palm, or the length of a foot or a pace, were objectionable chiefly on account of their lack of uniformity. Not only did different countries use different units for measuring quantities of the same kind, but even when a unit of the same name was used in different localities its value was not the same. In most civilized countries these older units have been entirely superseded by a new system of weights and measures, and in all countries this system has come into universal use for every kind of scientific work. It is usually called the C.G.S. System, from the initial letters of the units of length (the centimetre), of mass (the gram), and of time (the second). These three units are called *fundamental,* because they have been arbitrarily fixed in size, while all the other units of the system have been so chosen as to make them depend upon these three in as simple a manner as possible. For example, the *derived* unit of velocity is such as will denote movement through a single centimetre of distance in a single second of time, thus making the measure of a velocity numerically equal to the quotient of space traversed divided by time elapsed during the process. Similarly, the density of an object is defined as its mass in grams divided by its volume in cubic centimetres, so

that, although no name has been given to the unit, the density which is numerically equal to unity must be the density of such a substance as will weigh one gram for each cubic centimetre of its volume; the unit of force is the force that must act for one second of time in order to produce a change of one unit (one centimetre per second) in the velocity of a unit mass.

19. Unit of Length.—The scientific unit of length is the *centimetre*. It is equal to about half a finger-breadth and is often found on tape-measures, rulers, etc. These are simply copies of accurate standards belonging to the manufacturer, which in turn owe their accuracy to a careful comparison with the standards of the government. In the case of the governments that subscribed to the Metric Convention, including the United States, the standards, which are called *national prototype metres*, are lengths of one metre (*i. e.*, 100 centimetres) carefully laid off between lines near the ends of certain bars of artificially aged platinum-iridium alloy which are 102 centimetres long and have a cross-section that somewhat resembles a letter X (Fig. 1). The greater length is used instead of a single centimetre because it can be measured more accurately, and the cross-section is for the purpose of giving rigidity, and in order to allow the scale to be marked on a surface that would be neither stretched nor compressed if the bar should be slightly bent. The

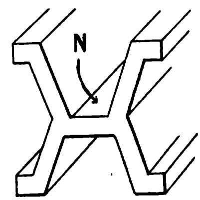


Fig. 1. Tresca Cross-section.—A bar of this shape has great rigidity for a given weight of metal; and a slight bending, such as would stretch the lower part and compress the upper part, has no effect on the "neutral web," *n*, where the scale is engraved. The diagram shows the exact size of the cross-section of the prototype metres.

standards were constructed at Paris and distributed by lot among the signatories to the Metric Convention about 1889, after being carefully tested and compared with one another so that their relative errors and equations were accurately known.* One of these, which is kept at the International Bureau of Weights and Measures, near Paris, is known as the *international prototype metre* and corresponds in length with the original flat platinum bar (100 cm. × 0.4 cm. × 2.5 cm.) constructed for the French Government by Borda and called the *mètre des archives*. The Borda standard was intended to equal one ten-millionth of the length of the meridian quadrant passing through Paris from the north pole to the equator, the earth itself thus furnishing the original standard. The metal bar, however, is now taken as the fundamental standard, not only because a microscopic measurement of it can be made more easily and more accurately than a geodetic survey, but also because the actual length of the earth's quadrant is not constant. Its average length, according to the best estimations, is about 10,002,100 metres.

The multiples and subdivisions of the metre that are in actual use are the kilometre (1 km. is 1000 metres, or about 5/8 of a mile; closer approximations are given in the appendix), the centimetre (1 cm.), the millimetre (0.1 cm.), and the micron (0.0001 cm.). These are more convenient than a single unit in some cases, but in scientific work it is desirable to express all lengths in terms of

* Modern processes of measurement are so accurate that a difference can generally be found between two standards that were intended to be equal, no matter how carefully they were constructed. The "equation" of a prototype metre expresses the way in which its length varies when its temperature is changed. Thus at any ordinary temperature, t, the length in centimetres of prototype metre No. 18 is $99.9999 + .0008642t + .000000100t^2$.

the accepted unit, 1 cm., in order to avoid possible confusion, or serious error in case the denomination of a quantity should be accidentally omitted. Thus, the length of the earth's quadrant is 1,000,210,000, but in order to be doubly safe it is advisable to make it a rule to *write the denomination after a number in all cases* (1,000,210,000 cm.).

20. Units of Area and Volume—The scientific unit of area is the *square centimetre* (1 cm^2), the area of a square each of whose sides is 1 cm. in length. A square foot is about 1000 cm^2. The unit of volume is the *cubic centimetre* (1 cm^3), the volume of a cube that measures 1 cm. on each edge. The dry and liquid quarts are each approximately equal to 1000 cm^3.

21. Units of Mass and Density.—The scientific unit of mass, or for practical purposes the unit of weight *in vacuo* at sea level, latitude 45°, is the *gram* (1 gm.), which is divided into 1000 milligrams (mgm.) just as the metre is divided into 1000 millimetres. It is derived from kilogram prototype standards (1 kgm. = 1000 gm. = 2.2 lbs.) established at the same time as the standards of length and was originally intended to be equal to the mass of one cubic centimetre of water under standard conditions. More careful measurements, however, on water that has been freed from dissolved air have shown that even at the temperature of its greatest density (3.98° C.) a gram of water occupies a trifle more space than one cubic centimetre, although the excess is only one sixtieth as great as it is at ordinary room temperature. In cases where the slight change of volume that is produced by heating or cooling can be neglected it may be considered that water has a density equal to one, *density* being defined as mass in grams divided by volume in cubic centimetres.

22. Unit of Time.—The scientific unit of time is the *second*, which is the 1/86400 part of the length of an average day from noon to noon. As the length of the solar day varies at different seasons of the year the second is determined in practice as 1/86164.1 of the time of a complete rotation of the earth with respect to the fixed stars. This unit of time was in use before the adoption of the C.G.S. System and is familiar to every one. It is perhaps worth noticing that fairly accurate seconds can be counted off by repeating, at ordinary conversational speed, "one thousand and one, one thousand and two, one thousand and three," etc.

23. Practice in Using the C.G.S. System.—The scientific system of units is so largely used that the student should not be satisfied with the mere ability to translate measurements from one system to the other. He should practice first estimating (guessing) and then measuring the dimensions of various objects that he comes across, until he has acquired a certain ability to "think" in centimetres, cubic centimetres, grams, etc., instead of in inches, quarts, and pounds.

Across the top of one page of the notebook draw a horizontal line just ten centimetres long and rule two short perpendicular lines across its ends in order to indicate the exact length clearly. In the same way, along the right-hand edge of the page, draw a line twenty-five centimetres in length. Under the first line draw five others of various lengths without measuring them. After they have been drawn measure each one with a scale of centimetres and millimetres, and record its length to the *nearest* millimetre. For example, if the length appears to be about $152\frac{1}{4}$ mm. it should be recorded as 15.2 cm.; if about $152\frac{3}{4}$ mm. it should be called 15.3 cm. Write each length as a number of *centimetres*, not 153 mm. nor

15 cm. 3 mm., and make it a rule to see that the denomination of a measurement is never omitted.

24. Rule for "Rounding Off" One Half.—It may happen that the measured length is so near 152½ mm. that it is impossible to decide between 152 and 153. A fraction perceptibly less than a half should be discarded and more than a half should always be considered as one more unit, but when it is uncertain which figure is the nearer one the universally adopted rule is to *record the nearest even number* rather than the odd number that is equally near. The reason for this procedure is that in a series of several measurements of the same quantity it will be as apt to make a record too large as it will to make one too small, and so in the average of several such values will cause but a slight error, if any. If the rule were that the half should be always increased to the next larger unit the errors would not balance one another and the average would tend to be brought up to a larger value than it should have. The same advantage would of course be obtained if the nearest odd number were always used, but the even number has one slight additional merit, namely, that in case it should have to be divided by two a recurrence of the same situation would be avoided.

By comparison with the lines already drawn make a mental estimate of the length and width of the notebook; then verify the estimate by measuring with a scale. Remember to record clearly all the experimental work that is done; thus, the completed notes should show at a glance which number is the actual width and which is the rough estimate.

25. The Hand as a Measure.—Lay your hand across a ruler or a metre stick and either spread the fingers slightly or crowd them closer together, as may be necessary, so

as to make a whole number of finger-breadths occupy the same amount of space as a whole number of centimetres. Then hold the hand in a similar manner while using it for practicing approximate measurements of various objects. Record also the exact measurements of the same objects as they are obtained later with the graduated scale.

Separate the thumb and little finger as far as can be conveniently done without special effort and measure your span in centimetres and millimetres. Repeat this measurement five times, being careful not to let the sight of the scale under your hand influence the extent to which the fingers are spread; and decide which is the most satisfactory value. Measure the length and the breadth of the table by means of successive spans, using hand-breadths or finger-breadths for the final fraction of a span, and compare the result with the actual length and breadth.

26. Measurement of Area.—Ask the instructor to draw an irregular outline in your notebook (Fig. 2).

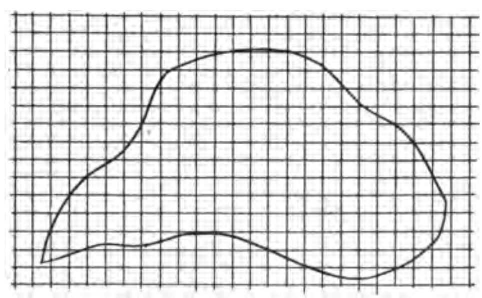


FIG. 2. IRREGULAR AREA.—The simplest method of measuring an area marked on squared paper is to count the squares that are entirely within the figure as units and those that are cut by the boundary as half units. The sum gives the total area almost as well as if an attempt were made to estimate the fractional size of each cut square.

Count the number of the small squares of the ruled paper which are entirely included within it, but do not outline them on the diagram. To their total add half the number of the squares that are cut by the boundary of the figure. The result will be the area of the irregular outline, not in square centimetres, of course, but in terms of the small ruled squares, and its denomination may be written "□'s." Try to obtain a more accurate value for the total area by estimating as closely as possible how many tenths of each cut square is included within the boundary and adding these actual fractions. The first result should agree quite closely with this one because any one of the fractions of a square is as likely to be less than one half as it is to be more than one half, and the most probable value for the average of the fractions is just one half.

As an alternative method, block off an equal area on the same irregular figure by drawing several rectangles and triangles to cover it (Fig. 3). If one corner of a

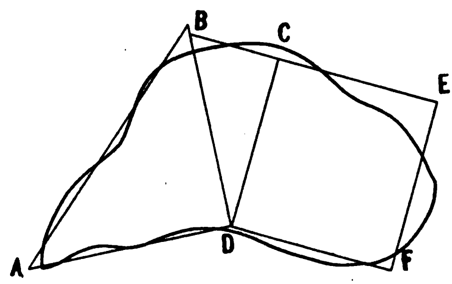


FIG. 3. AREA BY MENSURATION.—An irregular figure can be "blocked out" by a number of triangles and parallelograms which are so drawn that an error of excess in one place is approximately balanced by an error of defect in another, so as to make the combined areas of the geometrical figures equal to the required area. The geometrical areas are then evaluated by ordinary mensuration.

triangle projects considerably beyond the irregular line see that one of its sides is drawn so as to include less than the requisite amount and try to make the two opposite errors balance as nearly as possible. Then find the total area of the geometrical figures by mensuration, measuring their dimensions not by means of the centimetre scale but with a scale copied from the ruling of the notebook, or by transferring each length to the ruled page with a pair of dividers, so as to obtain the area in the same units as before.

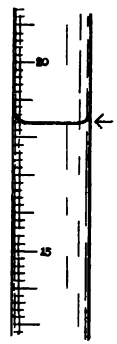


FIG. 4. GRADUATED CONTAINER.—The scale is customarily arranged to give the correct reading at the lowest part of the *meniscus*, or curved surface, of the liquid.

27. Measurement of Volume.—Examine the graduated cylinder and the graduated pipette and notice the volume occupied by one cubic centimetre in each. Do they seem larger or smaller than the space that would be enclosed if an imaginary square centimetre were drawn beside one linear centimetre of the ruler and an imaginary cube were then built up on the square? Pour into a test-tube an amount of water which you think will be 10 cm^3; then measure it carefully.*

Put an irregular block of metal (*not* one of the standard weights from your set) into a graduated cylinder partly filled with water and determine its volume by the change in the water level. In reading a cylinder or a pipette the result is to be obtained by noting the height of the liquid surface where it is lowest in the centre (Fig. 4).

28. Measurement of Mass.—Examine the brass

* See that a towel is at hand when water is used in any experiment, and wipe up immediately any that is spilled on the table.

weights in a set extending from one gram to 500 grams, and observe especially the size of the 10-gram weight. What would its volume be if its density were ten (gm. per cm^3)? If the density of brass is only 8.5 (*i. e.*, if it is less compact) will its volume be greater or less than this?

Examine the platform balance and notice that there are two wooden wedges that hold the pans away from the beam and the beam away from its support, so as to remove their weight from the accurately ground bearings when the apparatus is not in use. With a fine analytical balance this is such an important matter that a mechanism is provided by means of which the user keeps the scale pans and the beam "supported" while arranging the weights and the object to be weighed, and only lowers them upon their bearings for a few moments at the time of actual weighing. Such care is not necessary with an ordinary platform balance, but it is always advisable to support the scale pans after one has finished using the apparatus. Remove the wedges carefully and notice where the moving pointer comes to rest on the arbitrary scale. This need not be in the centre but may be at any point on the scale, and in weighing if the weights are so added as to return the pointer to this same position the result will be the same as if the pointer were made to take a central position in the first place. Notice the counterpoise, which can be screwed toward one side or the other in order to adjust the position of equilibrium, but do not attempt to move it unless you are sure that the apparatus is on a level part of the table and the scale pans are free from dust or other adherent matter. Notice whether the balance has a sliding weight that is used for weighing fractions of a gram.

Find the mass, in grams, of a four-ounce avoirdupois

weight, weighing it first on the left pan of the balance and then on the right. Notice that when it is on the right-hand pan the reading of the sliding weight must be subtracted from the sum of the other weights instead of being added to them.

Draw up a few cubic centimetres of water in the graduated pipette, retaining it by pressing the *dry* finger tip on the *upper* end of the tube. Practice letting it run out slowly until you have no trouble in delivering any exact amount. Then fill it again, weigh the "catch-bucket" or some other container, run just seven cm^3 of water from the pipette into the container and weigh the latter again.

29. Measurement of Density.—Calculate the density of water from the data obtained in the last experiment.

Find the volume of the 200-gram weight by measuring its height and diameter as carefully as possible, *but do not immerse it in water.* Suppose that the handle of the weight were soft, like wax, and could be flattened out and spread uniformly over the top of the cylindrical part, and estimate as well as you can how much this would add to the height of the cylinder. Then calculate the density of the brass weight, remembering that multiplication and division are always to be done by the abridged methods. The result may turn out to be less than the usual density of brass (about 8.5 gm. per cm^3) if the handle is a separate piece screwed into the body of the weight so that an air space is left between the two parts.

Find also the mass of the irregular solid whose volume was determined by immersion and calculate its density.

30. Equivalents.—In most English-speaking countries the C.G.S. System is very little used except for scientific

purposes, the older system still holding its own in spite of such obvious disadvantages as possessing an ounce (avoirdupois) that weighs about 28 gm. and another ounce (Troy) that is equal to a little more than 31 gm.; furthermore, the U. S. fluid ounce of water weighs more than an avoirdupois ounce and less than a Troy ounce, and is four percent larger than the imperial fluid ounce of England. Accordingly, a knowledge of the approximate relationships between the units of the old system and the new is almost a necessity for the scientific student of today.

Turn to the tables of equivalent weights and measures in the appendix of this book and use the *approximate* equivalents for translating your own weight and height into the C.G.S. System, and for answering such questions as: How many kilometres is the distance from here to New York (or any other city)? What is the approximate height of this room in metres? What is the C.G.S. velocity of sound if it travels a mile in five seconds? How many centimetres per second is your ordinary rate of walking?

31. Questions and Exercises.—1. Explain why rounding off a half to the nearest even number will increase a measurement, in the long run, just as often as it will decrease it.

2. When you drew a line "just ten" centimetres long was its length 10.0 cm.? Was it 10.00 cm.? 10.000 cm.? Measure it again, and explain why it is better to use one of these numbers in stating its length than to say "just" ten.

3. Does the sliding weight on the platform balance add its scale-indication to the right pan or to the left? How can it give correct results if the zero is not at the centre of its scale?

4. What disadvantage is there if the finger-tip that closes the top of the measuring pipette is not dry?

5. The *specific gravity* of a substance is defined as the ratio of its mass to the mass of an equal volume of water, *i. e.*, the *relative* density compared with water as a standard. How do the specific gravity and the density of a substance compare if the density of water is 1? If it is a little less than one?

6. If *specific volume*, v, is defined for any substance as the volume per gram of mass, what will the equation be that shows the relationship between ρ (density) and v?

7. If 10 cm. = 4 inches make a rapid mental calculation of the C.G.S. length of 12 inches. Of 40 inches; of 10 inches; of 3 feet; of 7 inches.

8. State the distance of 10 kilometres as the nearest whole number of miles. State 6 miles as the nearest whole number of kilometres.

9. Reduce 5 pounds per square foot (pressure) to gm. per cm^2. Reduce x lbs./ft.^{2} to gm./cm^2. Write in your notebook directions for reducing 1 gros (weight) per square pouce (length) to grams per square centimetre, or for reducing prices in roubles per pood to cents per kilogram, and submit them to your instructor for approval.

III. ANGLES AND CIRCULAR FUNCTIONS

Apparatus.—A pair of dividers; a pencil compass; a protractor; a ruler; a pencil with a fine point.

32. Unit of Angle.—When two straight lines intersect in such a way that each is perpendicular to the other the amount of their divergence is said to be ninety degrees or one right angle. A single degree is then a comparatively small difference in direction; two lines drawn from the observer toward the opposite edges of the sun's disc include an angle of about half a degree. For the sake of avoiding fractions at a time when decimals were not used, the degree (1°) was divided into sixty parts, each called one minute (1′), and a sixtieth of a minute was used as a still smaller unit, one second (1″). These three units are still in use all over the world for expressing the size of an angle as a "mixed" denominate number. For example the circular arc which is of the same length as the radius corresponds to an angle of 57° 17′ 45″.

33. Circular Measure.—If the original right angle had been divided and re-divided into tenths or hundredths instead of into ninetieths and sixtieths the resultant units would have been much more convenient for purposes of calculation. The question suggests itself, however, as to why the right angle should be arbitrarily chosen for the quantity that is to be subdivided. Why not take the whole circumference or some other amount of angle? As a matter of fact, this arbitrariness is often avoided for scientific purposes by measuring an angle by an entirely different method: The vertex of the angle is taken as a centre, around which an arc of a circle is drawn extending from one side of the angle to the other

(Fig. 5). The length of this arc will depend not only upon the size of the angle but also upon the length of the radius that is used, but the *ratio* of length of arc to length

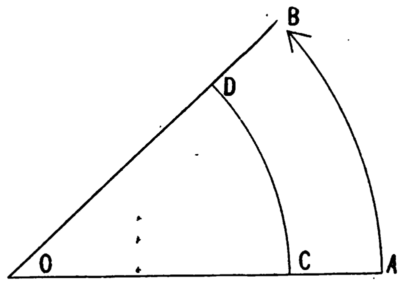


FIG. 5. CIRCULAR MEASURE.—The ratio of any arc to its radius is taken as the numerical or circular measure of the angle at the centre. $DC/OC = BA/OA = 3/4$, approximately, for the angle represented here.

of radius will depend only upon the size of the angle and so can be used as a measure of it. In the diagram the arc AB seems to be about $\frac{3}{4}$ as long as the radius OA, the arc CD is likewise $\frac{3}{4}$ as long as the radius OC, and the circular measure of this particular angle is accordingly $\frac{3}{4}$, or 0.75.

Draw an angle which is somewhat less than a right angle. Draw its arc, and measure the curved line as well as you can with an ordinary ruler. Measure the radius also, and calculate the circular measure of the angle. It will probably turn out to be about 1.4. Is the circular measure of a right angle equal to just 1.5? State why.

34. Numerical Measure of an Angle.—Since the size of an angle is defined as the quotient *arc divided by radius* it follows that this amount is not a number of centimetres or of any other arbitrary units but is a pure number. If the arc measures 6 centimetres and the

radius is 3 centimetres the size of the angle is the abstract number 2, not 2 centimetres; and if both had been measured in inches the quotient would still have been merely the number 2, the arc being *two* times the radius, not *two centimetres* times the radius. The expression *numerical measure of an angle* has the same meaning as *circular measure of an angle,* and denotes the way in which the size of an angle is always expressed for theoretical purposes. One of the chief advantages of this method lies in the simplification which it causes. Just as the foot-pound system of measures makes the density of water about 62 lb./ft³, and hence makes specific gravity approximately equal to *density divided by 62,* instead of merely to *density* as in the C.G.S. System, so angular velocity would be represented by 57.28 v/r instead of v/r, and such higher mathematical expressions as $D_x \sin x = \cos x$ and $D_x \cos x = -\sin x$ would become $D_x \sin x = .01746 \cos x$ and $D_x \cos x = -\pi \sin x/180$ if the angles were measured in degrees instead of numerically.

A suggestion of the reason why an angle ought to be expressed as an abstract number instead of in terms of a unit may be obtained by imagining a length of one centimeter and an angle of 45 degrees to be drawn on a sheet of paper and observed through a magnifying glass. The centimetre may appear to be enlarged to a length of two centimetres, but the angle of 45 degrees does not become an angle of 90 degrees; it remains exactly the same size as before. The same thing is of course true of an abstract number: with a very slight magnification two objects may both be made to look larger, but no amount of magnifying power will make them look like three.

35. The Angle π and the Unit Angle.—Draw an angle

of 180° and its arc, *viz.*, a semicircle. Obviously its numerical measure, semicircumference divided by semi-diameter, is the same fraction as the ratio of the whole circumference to the whole diameter, which is denoted by the symbol π and is approximately equal to 3.1416. It is also clear that an angle which extends entirely around a point, that is, four right angles or 360 degrees, must have 2π for its numerical measure; one right angle must be equal to $\pi/2$, $\pi/4$ is the same as 45 degrees, etc. If $180° = \pi$ the numerical measure of one degree must be $\pi/180$ and the degree-measure of an angle that is numerically equal to one must be $180/\pi$.

Find the number of degrees in the unit angle by writing the value of π, carried out to at least eight or ten decimal places, and using the method of abridged division to find out how many times it is contained in 180. The latter number should be written 180.00000... with as many ciphers as may be necessary.

Practice translating such numbers as the following into degrees until it can be done without any hestitation: $\frac{1}{2}\pi = ?$ $2\pi = ?$ $\frac{1}{4}\pi = ?$ $4\pi = ?$ $\pi/3 = ?$ $\frac{3}{2}\pi = ?$ $\pi/4 = ?$ $\pi/\pi = ?$ $\frac{1}{3}\pi = ?$ If an angle is numerically equal to three is it greater or less than 180°?

Practice translating the following numbers of degrees into circular measure until it can be done fluently: $90° = ?$ $360° = ?$ $45° = ?$ 3 right angles = ? $180° = ?$ $30° = ?$ $60° = ?$ $1° = ?$ $270° = ?$ $57°.296 = ?$

36. The Protractor.—A protractor is a scale of angles just as a graduated ruler is a scale of lengths. It consists essentially of a zero line on which is a point that represents the vertex of the angle, and a curved scale of short lines so placed that if they were sufficiently long each one would pass through the vertex and make an angle with the base line equal to the number of degrees with which it is marked.

Examine the protractor and its scale of degrees. Notice that the centre toward which the slanting lines converge must be on the line joining 0° and 180° and hence must be at the *top* of the notch shown in the figure. It is always at the corner that is made rectangular, the

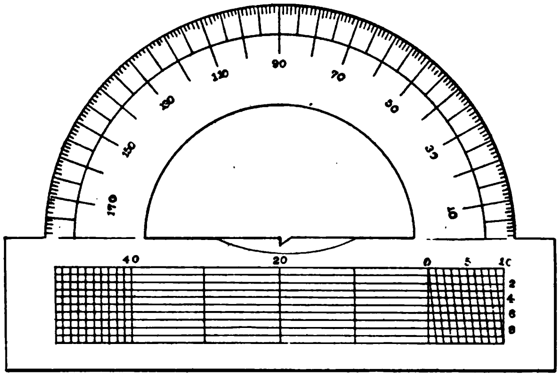


FIG. 6. PROTRACTOR.—The essential parts of a protractor are the zero line, the central point, and the scale showing where a line must pass to form the second side of any required angle.

other being obtuse or rounded. To draw a line which shall make a given angle with a given line at any particular point the protractor is placed so that its zero coincides with the given line and its centre with the given point. A fine line or dot is then made opposite the required number of degrees on the scale, the protractor is removed, and a ruler is used to draw a straight line through the dot and the particular point on the given line.

Use the protractor to draw a *triangle* of any convenient size and of such shape that its three angles are $\pi/2$, $\pi/3$, and $\pi/6$. Draw another so that its angles are $\pi/4$, $\pi/4$, and $\pi/2$.

37. The Diagonal Scale.—If the protractor is pro-

vided with what is known as a *diagonal scale* notice that at the top of this scale there is a horizontal line which is divided into centimetres or inches and that one division at the end of the scale is divided into tenths. Decide for yourself how it is that any length, such as 7.4, within the limits of the total length of the scale, can be found already laid off as a single continuous stretch of the base line, the tenths being measured from the proper point to the junction of the tenths' scale and the units' scale, and the units then extending onward the required distance beyond the junction point. Notice that the tenths' divisions are prolonged downward so as to cut diagonally across parallel horizontal lines and shift a single tenth to one side while dropping ten lines downward. This means that on the first level below the base line their shift will be only one hundredth, and on successive levels will be .02, .03, ...10, as the student can easily prove for himself by means of the principles of similar triangles. Accordingly, any number of units, tenths, and hundredths can be found marked off along the proper level. Thus a distance of 7.43 will be found on the third level between the same diagonal and vertical line as mark off 7.4 on the base line. A pair of dividers should be used to span an unknown length and then transfer it to the diagonal scale for measurement, or to take a required length from the scale for the purpose of laying it off on paper. They should be held rather flat against the scale, not perpendicularly, so as to avoid marring it.

38. Measures of Inclination.—If a slanting line intersects a level line the inclination of the former may be measured either by the size of the angle between them or by the rise of the inclined line per unit of level distance (see § 17). The second of these two amounts is said to be the *tangent* of the first one; thus $2/5 = \tan 22°$. This

relationship can easily be generalized so as to include cases in which neither side of the angle is horizontal: From any point on one side of an angle draw a straight line perpendicular to the other side; this will complete a right-angled triangle, and can be done in all cases where the angle is *acute* (*i. e.*, between 0° and 90° in size). The *tangent* of an acute angle of any right-angled triangle is defined as the ratio of the side opposite the angle to the side that is adjacent to the angle, the word "side" being used here to indicate either of the right-angled sides, not the hypothenuse.

In the diagram (Fig. 7) the angle G is an acute angle of

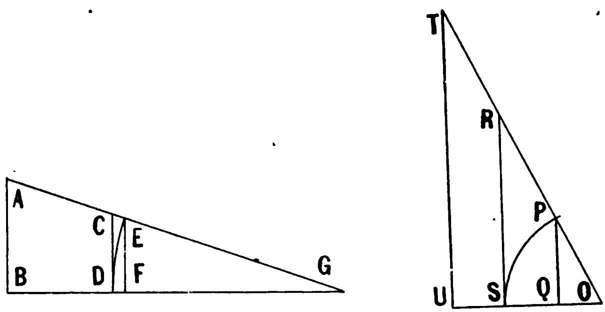


FIG. 7. ANGLES AND THEIR TANGENTS.—The ratio of any vertical side to the corresponding base is a number which is called the *tangent* of the acute angle at the base. EF is the same fraction of GF as AB is of GB, namely about 1/3. PQ is about twice OQ, and RS is about twice OS. Accordingly $\tan G = 1/3$ and $\tan O = 2$.

each of three different right-angled triangles. Measure the height and the base of any one of these and calculate the tangent of the angle G. The reason for the name "tangent" may be understood by noticing that the line drawn between the two sides of the angle and *tangent* to one end of the arc will be numerically equal to the ratio in question if the radius GD or GE is of unit length, for then $CD/GD = CD/1 = CD$.

The inclination of two lines that form an acute angle of a right-angled triangle may also be measured by the ratio of the opposite side to the hypothenuse. This is called the *sine* of the angle; for example, in Fig. 7, the ratio of EF to GE, or, what amounts to the same thing, EF/GD, is the sine of the angle G, often abbreviated to "sin G." It is evident that these definitions of *sine* and *tangent* agree with those that have been previously given for the "gradient" and "per cent slope" of an angle between a sloping line and a horizontal one.

39. Use of a Table of Tangents.—Close to the bottom of the next unused page of your notebook draw a fine horizontal line having a length of either 20, 25, or 50 of the squares made by the cross-lines of the paper. Draw it directly on one of the horizontally ruled lines, preferably so that it extends nearly across the page. Call its length unity (1); it may not be one foot or one decimetre, but it is to be considered as having a length of just one arbitrary unit, which need not be given any name. Mark a figure 1 under its right-hand end, a figure 0 at the left-hand end, and the scale of numbers 0.1, 0.2, 0.3, etc., at intervals of two, or two and a half, or five, squares, as may be required by the length of the line. From the right-hand end of this base line draw a fine line perpendicular to it, extending it as far as the top of the page. Beginning with zero at the junction with the base line lay off a similar scale along the vertical line. Do not number the successive squares of the paper 1, 2, 3, etc., but see that this scale indicates the same proportions as are given by the horizontal one. Turn to the table of circular functions in the appendix; look for the number 10 in the column headed DEG and notice that the number opposite it in the column TAN is 1763; this means that the tangent of 10° is .1763. On the

vertical line just drawn make a small mark at a height of .1763 above the base line according to the vertical scale already made. In the same way lay off tan 20° on the same scale. The decimal points are omitted from the table; but the tangent of a large angle is obviously greater than that of a smaller angle, and the table shows that for successive degrees the tangent increases gradually and quite regularly. Notice that the tangent of 45° is 1.000000..., and that angles and tangents larger than this must be sought for *above* the abbreviations DEG and TAN which are at the bottom of the page. If there is any trouble in understanding how the table is arranged use it to verify the following equations before proceeding further:

Tan 1° = .0175; tan 2° = .0349; tan 6° = .1051; tan 44° = .9657; tan 45° = 1.000; tan 46° = 1.036; tan 84° = 9.514; tan 85° = 11.43; tan 89° = 57.29.

Lay off tan 30°, tan 40°, tan 50°, etc., as far as the length of the vertical line will allow; then draw slanting lines from each of these points to the left-hand end of the base line. With the protractor test the angles formed in order to make sure that they are accurately 10°, 20°, 30°, etc. If mistakes have been made repeat the construction on the next page; do not correct the first diagram by erasures.

40. Experimental Determination of Sines.—With the base line as a radius and its left-hand end as a centre draw an arc on the diagram that has just been made, extending it from 0° to 90°. Complete the series of angles as far as 90° by laying off successive ten-degree arcs or chords with a pair of dividers. Find the point where the line whose slope is 10° intersects the arc, and note carefully the vertical distance from the base line up to this point, but do not draw a vertical line. If the diagram has been

carefully drawn with a sharp-pointed pencil the value should come out 0.174, being measured of course in terms of the figured scales, not in terms of the small ruled squares. Notice that this number is less than tan 10°, and that it corresponds to the ratio *EF/GD* in Fig. 7, and hence is the sine of the angle 10°. In the same way measure sin 20°, sin 30°, . . . sin 90°, or as many of them as may be directed by the instructor, and tabulate the results in the first two columns of a three-column table.

angle	*sine*
0°	0
10°	.175 − .001
20°	.345 − .003
30°	.500 + .000
40°	.640 + .003

Table of Sines.—The second column shows the measured value; the third column is the amount of change that must be made to obtain the true value.

Then turn to the table of circular functions, find the true numerical values of the sines from the column headed sin, and correct your measured sines by adding a third column as shown here; the sine of 40°, for example, appears to have been measured as .640 and then found to be actually .643, whence the correction of + .003 in the third column. After your table of sines has been completed find the angle whose sine has the largest correction and divide this correction mentally by the true value of the sine in order to find the relative error of the measurement. Thus, if the quotient is about 3/600 = 1/200 = .005 your measurement has an error that amounts to three parts out of a total of 600, which is the same as 5 per thousand, or ½ percent. Call your error "moderate" if it is anywhere between 0.3 percent and 1 percent; call it "large" if greater than 1 percent, and "small" if less than 0.3 percent.

41. Definition of Function.—Up to the present point it has been assumed that an angle of any size from 0 to

$\pi/2$ or 90° will have a tangent which is a definite number for each definite size of angle. A quantity which can be assumed to have different sizes, whether restricted to a certain range or not, is called a *variable;* and a second quantity, which in general has a definite value for each particular value of the first, is said to be a *function* of that variable. For example $x^2 - 3x + 2$ is called a function of x, because it has a definite value for any definite value that may be assigned to x. Similarly $\sin x$, $\tan x$, $\sqrt{x}$, logarithm of x, x^n, a^x, are all functions of x, as is also any other algebraical formula which involves x.

42. The Cosine of an Angle.—Another function of an angle which is frequently useful is the *cosine.* In a right triangle, such as was used in defining the tangent and the sine of an angle, it is the ratio of the adjacent side to the hypothenuse; the ratio of GF to GE, in Fig. 7, is the cosine of the angle G, or, as it is usually abbreviated, $\cos G = GF/GE$.

43. Circular Functions.—The sine, cosine, and tangent are included under the general term *circular functions,* a phrase which includes also three other functions of an angle which are not so frequently used. These are the *cotangent,* the *secant,* and the *cosecant,* and may be defined by the equations $\cot x = 1/\tan x$, $\sec x = 1/\cos x$, and $\operatorname{cosec} x = 1/\sin x$. Another set of definitions, which are perhaps more interesting and easier to remember, may be obtained by the use of a circle diagram and extended so as to include angles greater than 90°: For the sake of uniformity the angle is so placed that one of its sides extends out horizontally to the right. A zero angle would have its second side coincident with the first, and larger angles may be supposed to have been generated by rotating the second side through the required angular distance from the first. The usual convention is that

the direction of rotation shall be *counter-clockwise, i. e.*, in the opposite direction to that in which the hands of a clock turn. A circle whose radius is unity is drawn around the vertex of the angle, as in Fig. 8, which shows an angle of somewhat less than 45°. A line (DB) is drawn upward from the right-hand end (D) of the horizontal radius, and a perpendicular (AC) is dropped from the end of the inclined radius (A) to the base line (OD). Then the length of the tangent to the circle (DB) is the numerical tangent of the angle (O), the line (OB) that cuts the circle is the secant (Lat. *secans*, cutting), and the perpendicular

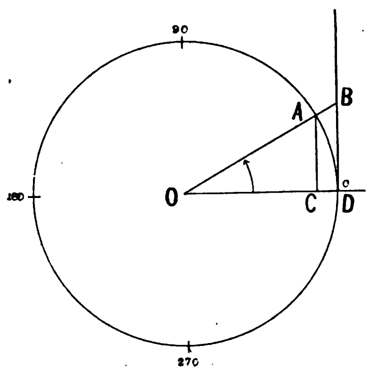


FIG. 8. CIRCULAR FUNCTIONS.— $DB = \tan\ DOA$; $AC = \sin\ DOA$; $OC = \cos\ DOA$; $OB = \sec\ DOA$. The radius is supposed to be of unit length.

(AC), which cuts off a rounded hollow of the figure, is called the sine (Lat. *sinus*, a bay). The cosine is the sine of the complementary angle (*e. g.*, $\angle\, OAC$; if x is acute $\cos x = \sin\,(90° - x)$), and the cotangent and cosecant are similarly the tangent and secant respectively of the complementary angle, two angles being called *complementary* when their sum is $\pi/2$ or 90°. As OC is considered to be a positive amount when measured to the right of O it is only natural to consider the cosine as a negative number when C is to the left of O. Likewise CA or DB must be negative if below the base line instead of above. AO and DO are to be produced if necessary.

If tan 45° = + 1 and sin 45° = + .707 what are the values of tan (90° + 45°) and sin (90° + 45°)? Ans.: Tan 135° = − 1; sin 135° = + 1.

What is the cosine of 135°? What are the tangent, sine, and cosine of 180° + 45°? Of 270° + 45°?

44. Generalized Idea of Angle. — For some purposes the term angle may be defined so as to include only angles less than $\pi/2$. For other purposes obtuse angles (*i. e.*, up to π or 180°) may need to be included. The circular functions have been already defined for angles of all sizes up to 2π (6.28, or 360°), but it is obviously unnecessary to stop at this figure. By supposing the line OA

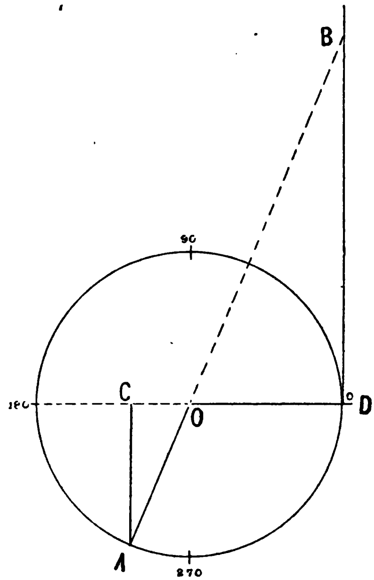


FIG. 9. FUNCTIONS OF ANY ANGLES.—The cosine is negative if C is to the left of O; the sine and tangent are negative if A and B are below the level of O; DO and AO are produced if necessary.

in Fig. 8 to have made more than a complete revolution it can be seen that the sine, cosine, tangent, etc., of 360° + 40° must all have the same values as the corresponding functions of 40°. In general, any circular function of any angle x is equal to the same function of $2\pi + x$, or of $4\pi + x$, or of $2n\pi + x$ if n is any whole number. A *negative angle* is of course one that is generated by a clockwise rotation from the position of the

base line; thus the angle shown in Fig. 9 may be considered either as $+ 240°$ or as $- 120°$, and obviously the circular functions of any negative angle $- x$ have the same values as those of the positive angle $2\pi - x$. Negative angles have to be considered occasionally, just as negative heights or lengths need to be. Angles larger than $+ 2\pi$ commonly come under consideration in connection with rotatory motion. Such objects as a spinning top, a fly-wheel, a planet, do not commonly move through an angle less than 360° and then stop, but their angular motion may be of almost any amount according to the extent of time occupied by it.

45. Questions and Exercises.—1. How can it be proved that the ratio which gives the numerical measure of an angle will have the same value whether the radius is long or short?

2. Translate 135° into circular measure. Make an approximate mental calculation of the number of degrees in an angle whose numerical measure is 6.

3. After drawing a triangle whose angles were meant to be $\pi/2$, $\pi/3$, and $\pi/6$ what would you do if you found one of its angles inaccurate but the other two correct?

4. The tangent of 80° is given in the table as 5671. Where do you place its decimal point and why?

5. The *steepness* of a slope is the characteristic which corresponds to its sine; a larger sine means a steeper slope. What characteristic can you name that will correspond to its cosine?

6. Any radius of a rotating wheel describes an angle which increases steadily from 0 to (say) 40π. Explain how the sine of this variable angle behaves during the same time.

7. What is the approximate value of tan $(- 91°)$? Of tan $(+ 89°)$? Of tan $(- 89°)$?

8. What is the approximate value of the tangent of the angle A in Fig. 7? How does it compare with the value of the tangent of the angle G? What relationship is there between tan T and tan O?

9. The *cotangent* of any angle, A, has been defined as the reciprocal of tan A, and also as the value of tan $(90° - A)$. Prove that these definitions are identical if $A < 90°$ by drawing a right triangle having A for one of its acute angles.

10. Draw a right triangle and prove that, if $A < 90°$, $\sin^2 A + \cos^2 A = 1$ using the theorem of Pythagoras that the square of the hypothenuse is equal to the sum of the squares of the other two sides.*

11. Make A an acute angle of a right triangle and prove that $\tan A = \sin A/\cos A$.

12. Using a circle diagram prove that in general for an angle of any size, x,

(a) $\tan x = \sin x/\cos x$

(b) $\sin^2 x + \cos^2 x = 1$

(c) $\tan (90° - x) = 1/\tan x$

(d) $\cos x = \sin (90° - x) = \sin (90° + x)$

13. What synonyms have you already learned for the sine and the tangent of an angle?

* $\sin^2 A$ is the conventional abbreviation of $(\sin A)^2$, not of $\sin (\sin A)$; similarly $\sin^3 A$, etc. $\sin^{-1} A$, however, is always used to denote *the angle whose sine is A*, not $1/\sin A$.

IV. SIGNIFICANT FIGURES

Apparatus.—Scale of centimetres and millimetres; card or strip of paper; circular brass measuring disc.

46. Estimation of Tenths.—It sometimes happens that a measurement requires but a slight degree of accuracy, and time and trouble can be saved by making it only roughly. As a general principle, however, it is advantageous to make all measurements as accurately as possible. Thus, measurements of length made with the metre stick should be expressed not merely to the nearest centimetre but to the nearest millimetre or tenth of a centimetre. This is not the best that can be done, however. After noticing how each centimetre of the scale is divided in half by a long line and each half is subdivided into fifths by four short lines, thus indicating tenths of a centimetre, it is not difficult to imagine each of the millimetre intervals of the scale divided in the same way into tenths of a millimetre and to make a fairly good estimate of just how many of these parts are included in the length that is to be measured. Experienced observers even attempt to make a mental subdivision into hundredths of the smallest intervals on a graduated scale, and find that it is only occasionally that one man's estimate will differ from another's by more than one or two hundredths, but for the beginner even the estimation of tenths will be a rather uncertain process. To gain proficiency it will be found better to begin with larger subdivisions, such as a scale of centimetres that has no millimetre marks. The position of a mark placed at random on such a scale can be estimated mentally and the accuracy of such a determination can then be tested

by actual measurement with a more finely divided scale.

47. Practice in Estimating Tenths.—Draw a short line at right angles to the edge of a card or slip of paper (Fig. 10) and hold this edge on a scale of centimetres and millimetres in such a way that the smallest graduations are hidden but the marks indicating centimetres and half-centimetres are visible. Notice the half-centimetre space in which the cross line comes and mentally divide it into five equal parts by four imaginary lines so as to make an estimate of the location of the line on the card. For example, in the figure the arrow seems to be either three fifths or four fifths of the way from the scale-division 19.5 to the division 20.0, making its position 19.8 or 19.9, but it may be difficult to decide which of these numbers is the nearer without actual measurement. In practice, however, the estimate should be written down and the card should then be allowed to slide carefully across the ruler until the millimetre scale is just exposed. The position on the scale which the line on the card occupies will then be ascertained and should be written in the notebook beside the previous estimate. The statement should be correct to the nearest millimetre, without any effort to decide upon fractions of a millimetre (see §§ 23 and 24).

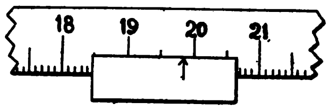


FIG. 10. ESTIMATING TENTHS OF A CENTIMETRE.—The card is laid on the scale at random, but in such a way as to hide the small dividing lines. The location of the arrow, in millimetres, is then guessed, and afterward verified by sliding the card downward enough to expose the whole scale.

Make ten such estimates with the card placed anywhere along the scale at random, and tabulate the determinations and the verifications in two parallel columns.

Then hold the card a trifle higher, so as to hide the half-centimetre graduations as well as the millimetres and make twenty more determinations. These will also be estimates of the nearest millimetre, but will require the more difficult process of deciding upon tenths of a whole centimetre instead of fifths of a half-centimetre.

If the scale that is used in the foregoing exercise is on the lower edge of a metre stick there will probably be a duplicate scale along its upper edge, with the help of which it would be easy to make each estimate absolutely correct before verifying it. This furnishes a good illustration of the statement that the student should aim to *learn* rather than to *do* (2). Finding out the correct position of a line over a hidden scale is something that is utterly useless to him or to anyone else; it is the learning to divide a centimetre into imagined tenths that will be valuable to him later when it becomes necessary for him to divide a millimetre into tenths without aid and without the possibility of later verification. In making any kind of a measurement care should be taken to avoid any extraneous influences, or bias or prejudice of any kind. In the present case, the upper scale on the metre stick should be kept covered in some way unless the student is sure of his ability to disregard it completely while making his estimations.

48. Mistakes in Estimating Tenths.—Omitting the preliminary estimate of fifths, examine your table of estimated tenths closely and find out what kind of error you are most apt to make. Some students find it hardest to estimate 0.3 and 0.7 correctly; others have almost a uniform tendency to read a position like 12.0 as either 11.9 or 12.1. The latter mistake is due to the fact that a minute deviation from the position of a visible graduation is very easily noticed and there is a tendency to consider

it as a single tenth. Of course if it amounts to more than half of a tenth this is correct; but if it is less than half a tenth it should be considered as 0.0 instead of 0.1. The same bias may even cause a tendency to read 0.1 and 0.9 as 0.2 and 0.8. On the other hand there may be just the opposite error if the graduated lines are rough or coarse, unless the observer is careful to estimate from the imaginary centre of such a line instead of from its margin.

If a definite kind of error is evident from a study of your table see if it can be overcome when making another short series of determinations. Then draw a second line on the card, place the latter in position as before, estimate both points, and find the distance between them by subtraction. This is the customary method of measuring a length, and is preferable to making one line coincide exactly with a scale division and estimating only the other one, in spite of an obvious additional source of error.

49. Value of π.—An experimental determination of the value of the constant, π, can be made by rolling the brass disc along a metre stick to find the length of its circumference, then measuring its diameter and calculating the ratio. Hold the disc loosely at its centre, using the thumb and forefinger only. Start it with its marked radius on some definite graduation of the scale and roll it in a straight line until the radius again comes vertically down on the scale. Read this second position, remembering not to be satisfied with the nearest millimetre (0.1 cm.) but to make as good an estimate as possible of tenths of a millimetre in order that the circumference may be correctly measured to a hundredth of a centimetre. Use the metre stick to measure the diameter of the disc with the greatest possible care, avoiding the end of the stick, which may be a little worn so that it does not represent

precisely 0.00 cm. or 100.00 cm. Find the value of π by dividing circumference by diameter, and in this particular case using the unabridged method of division and carrying out the result until it has two or three figures that are different from the theoretical value of π, which is 3.141592653589793238462643383279502884197169399. Finally round off (§§ 23; 24) both your result and the true value to the same number of places, choosing that number so that your result will show just one incorrect figure; for example, $3\frac{1}{7}$ (or 3.1428571) will have just one wrong figure if it is rounded off to 3.143, because the theoretical value will round off to 3.142; while 3.14234567 should be rounded off to 3.1423 to compare with the correct value 3.1416.

50. Physical Measurement.—The operation of making a measurement is merely counting; it is the determination of how many units of a certain kind are required in order to be equal to a given quantity of the same kind. But while a count such as a census of the number of individuals in a town must give a perfectly definite whole number it usually happens that a physical measurement will not give a whole number, or even a commensurable number except as the result of an error, and successive repetitions of a measurement will give a number of different apparent values. (Try it; measure the circumference of the π-disc a second time.) Accordingly, any numerical statement of a measurement must be merely an approximation to an unknown true value, and so will be either indistinguishably correct or perceptibly incorrect according to how closely it can be examined.

51. Ideal Accuracy.—The average student is liable to have more or less difficulty in grasping the idea that accuracy is always a relative matter and absolute precision of measurement is an impossibility. This is usually

because he has had very little practice in careful measurement and at the same time his previous study of arithmetic has emphasized a condition of infinite accuracy of numerical values. Such a number as 12.5 has been supposed not only to mean the same thing as 12.50 but also to be equal to 12.500000... to an unlimited number of decimal places. This is quite proper and satisfactory as long as one realizes that he is dealing with imaginary quantities, or perhaps it would be better to speak of them as ideal quantities, perfections of measurement which have no more reality of existence than the point, line, plane, or cube, of geometry. The smoothest surface of a table does not come as near to being a plane as does the surface of an "optically worked" block of glass or a "Whitworth plane," and even the smoothest possible surface can be magnified so as to show that it contains irregularities everywhere. Perhaps if it were magnified enough we could see that its shape would not even remain constant, but individual molecules would be found swinging back and forth or possibly escaping from the surface. A geometrical plane certainly corresponds to nothing in reality, and perfect accuracy of number is just as much an imaginary concept.

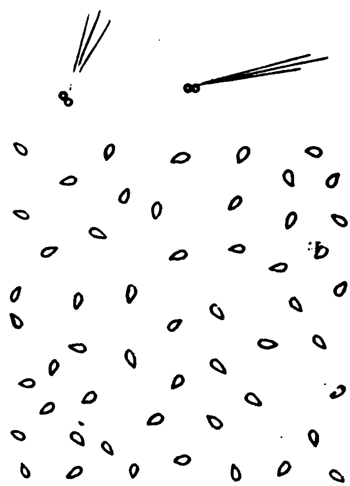


FIG. 11. DIAGRAMMATIC CROSS-SECTION.—A metal cube, greatly magnified, to show that there is no plane surface of contact between the metal and the air above it.

52. Decimal Accuracy.—If 12.5 cm., as a measurement, does not mean the ideal number 12.500000000... to an infinite number of decimal places what does it mean? As different measurements are likely to be made with different degrees of accuracy the universally adopted convention is merely the common-sense one that *the statement of a measurement must be accurate as far as it goes; and it should go far enough to express the accuracy of the determination.* Thus "12.8 cm." means a length that is nearer to precisely 12.800... than to precisely 12.7 or 12.9 cm., *i. e.*, that its "rounded-off" value would be 12.8 cm., not 12.7 or 12.9. If a length is written "12.80 cm.," however, this implies that the stated measurement is nearer to this same precise 12.8 or 12.80 or 12.800000... than it is to either 12.79 or 12.81 cm., in other words, that it has been measured to hundredths of a centimetre and found to be between $12.79\frac{1}{2}$ and $12.80\frac{1}{2}$ cm., so that it can properly be rounded off to 12.80. The other description, "12.8 cm.," means between $12.7\frac{1}{2}$ and $12.8\frac{1}{2}$; it states nothing about hundredths of a centimetre, and can correctly represent any lengths between the limits just given; for example 12.75, 12.76, 12.77, 12.78, 12.79, 12.80, 12.81, 12.82, 12.83, 12.84, or 12.85, for each one of these could be rounded off to 12.8. To write such a length of 12.8 cm. in the form "12.80 cm." would be to violate the rule that a statement should be accurate as far as it goes, for it would go as far as hundredths (stating that there were eighty of them), and the chances are ten to one that it would be one of the other numbers of hundredths given above. On the other hand, if an observer determined a length to be 12.80 cm., that is, if he measured the length as 12 cm. + 8 tenths + 0 hundredths—if he looked for hundredths and established the fact that

there were none of them,—then to state the measurement only as 12.8 cm. would not be doing justice to his own accuracy, for he would imply that the correct number of tenths was merely known to be nearer 8 than 7, namely greater than 7.5, whereas he had already found it to be nearer 8 than 7.9, namely greater than 7.95.

When a carpenter says "just 8 inches" he probably means "nearer to $8\frac{0}{8}$ than to $8\frac{1}{8}$ or $7\frac{7}{8}$ inches," a sixteenth of an inch one way or the other being unimportant. When a machinist says "just 8 inches" he may mean "nearer to $8\frac{0}{64}$ than to $7\frac{63}{64}$ or to $8\frac{1}{64}$," a half-sixty-fourth or hundred-and-twenty-eighth of an inch being negligible to him. When another person says "just 8 inches" we must know what kind of materials he works with before we can tell the meaning of his word "just." If decimal subdivisions were everywhere used the carpenter's eight inches would probably mean 8.0 while the machinist's would mean 8.00; for one man "8" would mean "between 7.950 and 8.050" while for the other it would mean "between 7.995 and 8.005." It is for the sake of avoiding such ambiguities that the scientist has adopted the rule that "8" means "between 7.500 and 8.500"; "8.0" means "between 7.950 and 8.050"; "8.00" means "between 7.995 and 8.005"; "8.000" means "between $7.999\frac{1}{2}$ and $8.000\frac{1}{2}$"; etc.; in other words: *no more figures are to be written down than are known to be correct; and, no figures that are known to be correct should be omitted.* This principle is simple enough when it has once been properly comprehended, and after that there is not much danger of the student's "rounding off" a carefully obtained measurement like 2.836 gm. to 2.84 gm. merely for the sake of doing some rounding off. There is a very decided likelihood, however, that he will often forget to write down a final significant zero; if two lengths are

147 mm. and 160 mm. the tendency when writing them in centimetres is to put down 14.7 and 16. If the zero is as important as the seven when writing millimetres the same is equally true when writing centimetres. Suppose the diameter of the π-disc is found to be "just 8" centimetres; the measurement should be stated as 8.00 cm. if tenths of a millimetre were estimated and none were found, but it should be given as 8.0 cm. if the student read the millimetres but was unable to make an estimate of smaller amounts. There is nothing to show which degree of accuracy was obtained if the diameter is put down as 8 cm. "because it came out just even."

53. Significant Figures.—In the expression 6.2 cm. both the figure 6 and the figure 2 mean something or are *significant.* In the expression 62 mm. there are likewise two significant figures. When the same length is written in the form .062 metre there is no difference in what is signified, and although the number has three figures it is still said to have only two significant figures; the zero is present merely for the purpose of showing which decimal places are occupied by the six and the two, or, in other words, for fixing the location of the decimal point. If the same length is called 62 thousands of micra or 62000 μ there are still only two significant figures, and again the ciphers serve only to show that the six is located in tens-of-thousands' place, *i. e.*, to fix the position of the decimal point. *In arabic notation, the figures of which a number is composed, except for one or more consecutive ciphers placed at its beginning or end for the purpose of locating the decimal point, are called its significant figures.* In accordance with this definition it will be clear that only two of the three figures of 0.75 gm. are significant, and only one of the two figures of such an expression as 05c., in which a superfluous zero

is sometimes written. Non-significant ciphers occur sometimes on the left, as in the statement that a certain light-wave has a length of .00005086 cm., and sometimes on the right, as when the sun is stated to be 93000000 miles from the earth; the first of these numbers has four significant figures, the second has only two. It will be noticed that a number like the last causes trouble in applying the rule of making it "accurate as far as it goes," for only the first two or three figures are known, but eight are needed in order to place the decimal point. A further source of trouble lies in the fact that the last figure which is significant may happen to be a cipher instead of some other digit. If this is to the right of the decimal point the zero is of course written when significant and omitted when not (as explained in § 52), but what is to be done when a similar case occurs in which the significant zero occurs to the left of the decimal point? If a building is said to be worth fourteen thousand dollars how is any one to tell whether this means 14 thousands of dollars, *i. e.*, nearer to 14 than to 13 or 15 of these thousands, or whether it means exactly 14000 dollars and no cents, or whether the number of significant figures is not intended to be either two or seven but some intermediate number? Suppose the number of millions of miles from the earth to the sun at some particular time is found to be 93.00; we need two different symbols for our ciphers so that we can write 93,00o,ooo miles to show that the first two ciphers are significant while the last four are not. There is really no reason for using ciphers at all in the last four places, except that it is customary, and it would be better to use some other character, such as ɛ, and write 9300ɛɛɛɛ. Neither of these methods is ever used, however, but the same result is achieved by a notation that will be explained later on.

The length of an inch has been determined to be between 25.39977 and 25.39978 mm. To state that it is 25.40 mm. is correct, because this value is accurate as far as it goes; but to say that 1 inch = 25.39 mm. would be wrong, for the true number of hundredths is nearer to 40 than to 39. If it is desirable to use an approximate value, so that rounding off is permissible, how many of the following statements are correct and how many are positively wrong? 1 in. = 25.4 mm.; 1 in. = 25.40 mm.; 1 in. = 25.400 mm.; 1 in. = 25.4000 mm.

Which of the following values of π are correct, and which are incorrect? 3.141592; 3.141593; 3.141600; 3.14160; 3.1416; 3.1415; 3.142; 3.141; 3.15; 3.1;

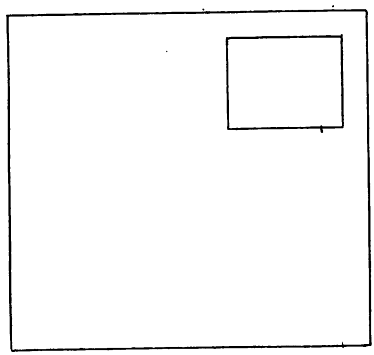


FIG. 12. RELATIVE AND ABSOLUTE ACCURACY.—The large rectangle comes nearer to the shape of a perfect square than the small one does, although the difference in its two dimensions is precisely the same as the difference between the height and the width of the small rectangle. Accuracy or inaccuracy can be considered great or small only relatively to the size of the quantity that is being measured.

3.142857 (= 22/7); 3. (II Chronicles, chap. 4, v. 2); 3.16 (= $\sqrt{10}$).

At ordinary room temperature (20° C.) is the density

of water equal to 1? Is it 1.0? 1.00? 1.000? To what point must its temperature be lowered in order to make its density 1.0000? (See tables; appendix.)

54. Relative Accuracy.—The diagram (Fig. 12) shows two rectangles which are approximately squares. The difference between the height and the width of the larger one is just the same as the difference between the height and the width of the smaller, yet the small rectangle is obviously a less accurate approximation to the shape of a perfect square than is the large one. This may serve as an illustration of the important general statement that accuracy is a matter of relative amount rather than of absolute amount. A sixteenth of an inch has the same absolute value wherever it occurs, but it is a considerable part of a quarter-inch length while it is relatively insignificant in comparison with a whole inch.

The relative accuracy of a measurement accordingly depends upon two things: how much its absolute difference from the truth amounts to, and how large the measurement itself is. If two points on the earth's surface are found by careful surveying to be 10 miles apart the determination of distance may easily be in error by more than a foot, and even with the most extremely careful triangulation the error is likely to be as much as four inches. An error of a quarter of an inch, however, in measuring the thickness of a door could hardly be made even with the clumsiest of measuring apparatus. It would plainly be misleading to say that the clumsy measurement should be considered more accurate than the careful one on account of $\frac{1}{4}$ inch being less than 4 inches. The only consistent way of looking at the matter is to inquire how large a fraction of the total measurement the error amounts to. Suppose the thickness of the door is $1\frac{1}{2}$ inches; how large a part of this

measurement is the error of $\frac{1}{4}$ inch? Obviously it is one sixth of the total or an error of more than 16 percent; while four inches out of a total of ten miles is not nearly a sixth, but is roughly an error of one out of a hundred and fifty thousand, or about six per million, or about .0006 of 1 percent.

How large an error is an eighth of an inch when half of an inch is being measured (Fig. 12)? How large is a sixteenth of an inch out of a total of an inch and three quarters?

If a measurement is stated to be 12.8 cm. when it is known to be between 12.750 and 12.850 cm. what is the greatest possible error of the statement? Answer: .05 out of a total of 12.75. This is the same as 05. out of 1275., or 5 per 1275, or 1 per 255; 1/250 would be 4/1000 or .004, so 1/255 must be a little less than .004, *i. e.*, a little less than 0.4 percent.

55. Calculation of Relative Errors.—The relative error of a measurement does not usually need to be calculated with any very great care. Where numbers are as different as 6 in tens-of-thousandths' place (10-mile survey, above) and 16 in units' place (thickness of door) the location of the decimal point is really more important than the size of the significant figure that occupies either place; to call the former number "a few ten-thousandths of a per-cent" and the latter "some ten or twenty percent" gives all the information that is needed. This means that a calculation of relative error never needs to be done on paper but can always be worked out as a rough mental calculation. Thus, in the illustration given above, 1 foot is 1/5280 of 1 mile, hence it is 1/52800 of 10 miles or roughly about 1/50,000; and 4 inches, being $\frac{1}{3}$ of 1 foot, is $\frac{1}{3}$/50,000 of the whole distance, or 1/150,000. The denominator of this fraction is about a sixth of a million

(since 15 is about $\frac{1}{6}$ of 100), so 1/150,000 = 6/1,000,000 = .000006 = .0006 percent.

Decide mentally what percent .01 is of 7.23. Ans.: .0015, or .15 per cent.

What percent of 94.07 is .01?

56. " Decimal Places " versus " Significant Figures."—If a length is stated as 174.2 cm. the inference is that it is nearer to that exact amount than to 174.1 or 174.3 cm., namely that its error certainly is not as much as 0.1 out of 174.2. This is the same as saying that it is not as much as 1 out of 1742, or 1 out of nearly 2000, or 5 per 10000, or .0005, or .05 percent.

In the following table the left-hand column contains five numbers, all of which are carried out to the same number of decimal places; namely, two. In the right-hand column notice that the same five numbers occur, but each one of them is carried out to the same number (3) of significant figures, no matter how many decimal places there may be. If the accuracy of each of these ten numbers should be worked out in the way that has just been explained, would the numbers in the left-hand column turn out to be all of approximately equal accuracy while the right-hand column showed great accuracy for one number and little for another, or would the numbers on the right be the ones that would be about equally accurate while those on the left fluctuated? In other words, is accuracy a matter of decimal places or of significant figures?

7.23	7.23
94.07	94.1
0.52	0.522
428.00	428.
66.67	66.7

TABLE OF FIVE NUMBERS.—In the left-hand column each number has two decimal places but some have more significant figures than others. In the right-hand column, each number has three significant figures but some have more decimal places than others.

The table has been repeated on this page and shows the answer to the question. Beside each of the ten numbers its accuracy has been written down as a percentage, and it will be seen that the numbers in the right-hand column all show about the same degree of accuracy, while those in the left-hand column differ widely.

.1%	7.23	7.23	.1%
.01%	94.07	94.1	.1%
2. %	0.52	0.522	.2%
.002%	428.00	428.	.2%
.01%	66.67	66.7	.1%

Table Showing Accuracy of Numbers.—Notice that the number of decimal places to which a measurement is carried out has nothing to do with its accuracy. It is the *number of significant figures* that determines the matter.

Turn to the table in the appendix where errors are classified according to their size and write the appropriate word opposite each of the percentages given in the right-hand column of the table on this page. Then do the same way with those in the left-hand column.

57. Rule for the Relative Difference of Two Measurements.—The difference between 3.11 and π ($= 3.14$) is 3 out of 314, and the difference between 3.17 and π is likewise 3/314, not 3/317; *i. e.*, the numerical error is to be divided by the true or theoretical value rather than by the experimental or erroneous value. It is often desirable, however, to compare two values which are equally good, according to one's available knowledge of them. When there is no standard and no reason for choosing one of the measurements rather than the other the accepted procedure is to *divide the difference by the greater value.* For example, the numbers 4 and 5 would be said to differ from each other by 20 percent, not 25 percent, for the difference divided by the greater number

is one fifth, not one fourth. In cases of fairly accurate measurements it is unimportant whether the larger number or the smaller one is taken for the divisor, but for the sake of uniformity it is customary to choose the larger one.

Apply this rule to your two measurements of an irregular area (§ 26), obtained by counting squares and by constructing geometrical figures. How much relative difference is there in the results of the two methods?

58. Accuracy of a Calculated Result.—Multiply 65.97 by 24.15, using the abridged method of multiplication. Compare your product with that of example (*c*), § 16. It will be noticed that changing the fourth figure of one factor has produced a change in the fourth figure of the product. This means that if only three figures of the factor had been known, the fourth being uncertain, no calculation could have given more than three trustworthy figures of the product, because the fourth figure would have depended upon the unknown fourth figure of one of the factors. Likewise, in division, if only five figures are known of either the divisor or the dividend there is nothing to be gained by keeping more than five figures in the other one; only five figures of the quotient will have any meaning, and if further figures are obtained by any process of calculation they will be unjustified and misleading. The general rule will be obvious: the result of a multiplication or division will have no greater accuracy than that of the least accurate of the data from which it is obtained.

59. Accuracy of the Abridged Methods.—Remembering that the accuracy with which a quantity is expressed depends not upon the number of decimal places but upon the number of significant figures and keeping in mind the fact that the number of trustworthy figures in a

product is the same as the number in its least accurate factor, turn back to your notes on §§ 15 and 16 and observe that the method of abridged division automatically gives just the number of figures in the quotient that are needed if no figures of the dividend are "brought down"; and that abridged multiplication always gives at least as many as are in the shortest factor. It will not give any superfluous figures if the longer factor is used as the multiplier, but will give as many as the longer factor contains if that is used as the multiplicand. Of course the best method is to round off the longer number before beginning the calculation, so that it has no more figures than the shorter one.

60. Standard Form.—To avoid a long string of figures when writing very large or very small numbers it is customary to divide a number into two factors, one of them being a power of ten. Thus, .00000017 and 632000000000 are the same as 17×10^{-8} and 632×10^{9} respectively. This notation also makes it possible to write 93000000 unequivocally with either two significant figures or four, as may be desired (see § 53), for it can be put either in the form 9.3×10^{7} or in the form 9.300×10^{7}. The same value and accuracy for 9.300×10^{7} would be retained just as well by writing 93.00×10^{6} or 930.0×10^{5}, but it is customary to choose the power of ten so that the other factor shall have just one significant figure to the left of the decimal point. The number is then said to be written in *standard form.*

Write the following numbers in standard form: 2946.3; 632×10^{9} (ans.: 6.32×10^{11}); 17×10^{-8}; 25.39978; 0.0073; .007300; 666.6; .001; .0010; 107.42; 186000; 2.5400; 3.1416; 9.9942; 2.54ƐƐ. Prove that each result is correct by performing the indicated multiplication.

Write a definition of standard form in your own words.

61. Questions and Exercises.—1. What precautions did you take in order to measure the diameter of the brass disc with the greatest possible accuracy?

2. Write your measured value of π, carrying it out just far enough to show one wrong figure. How many significant figures of 3.141625 are correct? Of 3.1424?

3. State some of the possible causes that make your determination of π incorrect. Would there be any advantage in taking the average of several measurements of the circumference? In rolling the disc through two or more consecutive revolutions and measuring the total distance?

4. Is there any difference in meaning between the italicized statement at the beginning of § 52 and the one near the end of the same section? If so, what?

5. How many significant figures do you think there are in the length of the earth's quadrant as given in § 19?

6. Show that the last statement in § 57 is correct by taking some measurement that you have made as an example.

7. If one gram is equal to 15.432 grains how much is five grams? If a weight of five grams is the same as 77.16 grains how much is one gram? (The answers 77.16 grains and 15.432 grains are both wrong.)

8. How is it that a metre can be measured more accurately (§ 19) than a centimetre?

9. Each side of a square measures 82.5 mm. How many centimetres long is its entire periphery? ("Just 33 cm." is not the correct answer.)

10. Use the abridged method for multiplying 12 by 13, and for multiplying 13 by 12. Why does the answer come out 16 each time? How should it be pointed off? How many of its figures are significant?

If you wanted to obtain three significant figures in the product, using the abridged method, what would you do? (Answer: Use three significant figures for both factors, 12.0 and 13.0. Try it.)

11. In exercise 9 how many significant figures did you keep in the product of 82.5 × 4? How many are you entitled to keep? Does the 82.5 mean the same as 82.50? Does the 4 mean the same as 4.0? As 4.000?

12. Turn to the table in the appendix where the density of water is given. Does water at a temperature of 4° C. have a density of 1? Of 1.000? Of 1.0000? Of 1.00000? Correct the following statement by crossing off the unjustifiable figures, but keep all that are correct: "at ordinary room temperatures water has density of 1.00000."

13. In § 35 why were you justified in adding ciphers *ad libitum* to the number 180?

V. LOGARITHMS

62. Definitions.—The *logarithm* of a number is defined as the *power* to which ten or some other numerical quantity must be raised in order to give the specified number. Thus, the logarithm of a thousand is 3 and the logarithm of a hundred is 2, for $1000 = 10^3$ and $100 = 10^2$. These statements are usually abbreviated to "log 1000 = 3" and "log 100 = 2." The number which is raised to some power is called the *base,* and logarithms which have 10 for a base are called *common logarithms.* For theoretical purposes what are known as *natural logarithms* are often used; their base is a number which is denoted by the letter e (approximately 2.71828) and is equal to the infinite series $1+1+1/2+1/2\cdot3+1/2\cdot3\cdot4+1/2\cdot3\cdot4\cdot5 + \ldots$ or to the limit of $[(1 + (1/n)]^n$ when n is increased indefinitely. To avoid confusion the base is often written as a subscript; thus, $\log_{10} 100 = 2$ and $\log_e 100 = 4.6052$ mean exactly the same thing as $10^2 = 100$ and $e^{4.6052} = 100$. The only logarithms that will be considered here are those whose base is 10.

63. Fundamental Properties of Logarithms.—The table on the next page gives the values of various integral powers of ten; in other words it gives the numbers which have integers for their logarithms. Pick out any two logarithms (exponents) and add them. Then notice that the sum which is thus obtained is another logarithm, namely, the logarithm of the product of the two numbers that correspond to the original logarithms. For example the logarithm of 100 is 2, and of a thousand is 3; adding them, 5 will be found from the table to be the logarithm, not of the sum of $100 + 1000$, but of the product

100 × 1000, or 100000. One of the chief uses of logarithms is to enable a multiplication to be performed by the simpler process of addition. In the particular case just given it is as easy to multiply 100 by 1000 directly as it is to add their logarithms and see what number corresponds to the sum, but an exercise like 6.28 × 17.35 is as easy as 100 × 1000 when worked out by logarithms although it would mean much more time and trouble to multiply it out, even if the abridged method were used. The process is simply to add log 6.28 to log 17.35 and the result will be the logarithm of their product, 6.28 × 17.35. In general,

$$10^6 = 1000000$$
$$10^5 = 100000$$
$$10^4 = 10000$$
$$10^3 = 1000$$
$$10^2 = 100$$
$$10^1 = 10$$
$$10^0 = 1$$
$$10^{-1} = .1$$
$$10^{-2} = .01$$
$$10^{-3} = .001$$

TABLE OF COMMON LOGARITHMS. — The exponent of the base, 10, on the left, is called the *logarithm* of the 'natural number' on the right.

$$\log a + \log b = \log (a \times b). \tag{1}$$

Try numerical values for the following also, taking each equation separately in turn, and extending the above table if necessary:

$$\log a - \log b = \log (a \div b), \tag{2}$$

$$n \times \log a = \log a^n, \tag{3}$$

$$(\log a) \div n = \log \sqrt[n]{a}. \tag{4}$$

These four equations give the fundamental principles involved in the use of logarithms. The student should not attempt to memorize them as equations, but will need to be perfectly familiar with the ideas that they express. Notice that when using logarithms addition takes the place of multiplication, subtraction of division, multi-

plication of raising to a power, and division of root extraction. Addition and subtraction are performed on logarithms; multiplication and division are also performed upon logarithms but the multiplier or divisor is the number itself (*natural number,* as it is often called to distinguish it from the logarithm or *logarithmic number*), not the logarithm of the number. The result in all cases is a logarithm, and from this the required number is found by consulting a table.

64. Common logarithms.—The advantage of using 10 as a base is that

$$\begin{aligned} \log (10 \times a) &= \log 10 + \log a \qquad \text{(from eq. 1)} \\ &= 1 + \log a \end{aligned}$$

and in general

$$\begin{aligned} \log (10^n \times a) &= n \log 10 + \log a \qquad \text{(from equation 3)} \\ &= n + \log a; \end{aligned}$$

for example, $\log 365 = \log (10^2 \times 3.65) = 2 + \log 3.65$. Accordingly tables of common logarithms are made out only for natural numbers between 1 and 10, the logarithms of all other numbers being self-evident from these.

nat. no.	*log.*
1	.000
2	.301
3	.477
4	.602
5	.699
6	.778
7	.845
8	.903
9	.954
10	1.000

TABLE OF LOGARITHMS. — The logarithms of the natural numbers from 1 to 10 are given here as far as the first three decimal places.

If the logarithm of 3.65 is .562 what is the logarithm of 3650? Ans.: 3.562. What is log 365? Log 36.5? Log .365? Ans.: $-1 + .562$. Log .0000365? Ans.: $-5 + .562$. (Do not simplify these binomial forms. They are easier to use if left as they are.)

The logarithms of numbers other than powers of 10 are in general incommensurable and

are given only approximately in tables. Use only the small table given on the last page for the exercises in the following paragraph.

Find 2×3. Answer: log 2 = .301; log 3 = .477; their sum is .778, and by looking in the table this is found to be the logarithm that corresponds to the number 6. (Six is sometimes said to be the *anti-logarithm* of .778.) Using logarithms, find 2×4. (Do *not* perform the multiplication mentally and then look for the logarithm of 8 to verify the sum of log 2 and log 4, but consider the product as being unknown until after you have been directly led to it by following out the logarithmic process.) Find 2^2; find 3^2. Find 4×5; $\sqrt{9}$; 5×6; 50×6; 500×600.

Calculate the value of e from the infinite series given above.

65. Practical Logarithm Tables.—Examine the four-place logarithm table in the appendix at the end of this book, and notice that it contains the same succession of numbers, from 1 to 9, as the small table which has just been used. It also contains the same succession of logarithms, from .0 to .9; but the intermediate values, both of logarithms (.0000 to .9999) and of natural numbers (1.00 to 9.99), are given at smaller intervals, and without any decimal points. Verify each of the following statements by finding the required logarithm *in line* with the first two figures of the natural number as they occur in the left-hand column, and in the *column* that is headed by the third figure: log 3.65 = .5623; log 3.66 = .5635; log 4.06 = .6085; log 7.70 = .8865; log 77.0 = 1.8865 (§ 64); log 7700 = 3.8865; log .00077 = − 4 + .8865.

Find the logarithms of 5.02; 5.01; 5.00; 50.0; 5000000.

It will have been noticed that the decimal part of

a logarithm (sometimes called the *mantissa*) is dependent only upon the arrangement of signicant figures in the natural number; *e. g.*, log 36500 = 4.5623; log .0365 = $-2 + .5623$. The integral part (sometimes called the *characteristic* of the logarithm), however, is determined only by the position of the first significant figure; for example, for any number beginning in tens-of-thousands' place it is 4, and for any number beginning in hundredths' place it is -2. In order to save space a number like log .0365 is customarily written in the form $\bar{2}.5623$, the minus sign being written *over* the characteristic to indicate that it applies only to the whole number while the decimal part is always positive.

Write -2.60 in logarithmic form. Ans.: $-2.60 = -2 - .60 = -3 + .40 = \bar{3}.40$. Write -1.4377 with the decimal part positive. Ans.: $\bar{2}.5623$.

Write the characteristic of the logarithm of each of the following: 5441; 27; 79264; 264; 73; 0.73; 0.073; 0.000073. Make up a rule for finding the characteristic of the logarithm of any number, and write it in your notebook.

Write the logarithms of 984; 982; 981; 980; 98; 9.8; .98; .098; 7; 14. Add the last two and find their sum in the body of the table; see what number in the margin corresponds to it, and verify the result by multiplying 14 by 7.

66. Use of the Table.—A four-place table of logarithms is in general satisfactory for obtaining the logarithm of a number that has four significant figures; for numbers of three significant figures it is not necessary to keep more than three decimal places of the logarithms; for five-figure accuracy a five-place table is needed; etc. A four-place table is generally made more compact by including only three-figure values for the natural numbers, and when the logarithm of a four-figure value is

required it is found by a process called *interpolation,* in which it is assumed that small differences between logarithms are proportional to the corresponding differences in their antilogarithms. Turn to the table and notice that (log 633 − log 632) is exactly one third as large as (log 635 − log 632); and of course the differences in the numbers 633 − 632 and 635 − 632 are in the same ratio. Suppose the logarithm of 3.142 is required: The table gives log 3.14 and log 3.15. The required number 3.142 is larger than 3.140 but smaller than 3.150; in an orderly scale of numbers it would be located *just one fifth of the way from 3.14 to 3.15.* The assumption is, accordingly, that its logarithm *likewise is situated one fifth of the way* from log 3.14 to log 3.15. Log 3.14 = .4969; log 3.15 = .4983. One fifth of the distance from .4969 to .4983 is obtained by first finding that distance—subtracting; 4983 − 4969 = 14;—then adding the required fraction of it to the lower logarithm—1/5 of 14 = 3 (the nearest whole number), and 4969 + 3 = 4972; ∴ log 3.142 = .4972.

Suppose the number is required whose logarithm is .2752: Turn to the table and notice that the nearest logarithms are 2742 and 2765. Their difference (called the *tabular difference*) is 23, and the given logarithm 2752 is 10 larger than 2742; *i. e.*, it lies 10/23 of the way from 2742 to 2765. Accordingly the required antilogarithm will be 10/23 of the way from one marginal number (188) to the other (189); that is, it will be $188\frac{10}{23}$, or 188.4. As the characteristic of the given logarithm is zero this should be pointed off as 1.884.

The small multiplication tables at the side of the logarithmic table enable a fraction like 10/23 to be reduced to tenths mentally. Find the table headed 23 and notice that it gives one tenth of 23 = 2.3; .2 of

23 = 4.6; etc. The number nearest to 10 is 9.2, which stands opposite 4; accordingly 10 twenty-thirds comes nearer to 4 tenths than to 5 tenths. Where 1/5 of 14 was required, above, the small tables would have been used, if necessary, by finding the number which is opposite 2 tenths in the fourteen table.

Find log 2.718. Ans. 0.4343. Find log 3.333; log 1.234; log 12.34; log 123.4; log 8888; log .4343; log 3449.

67. The Probability Function.—In much of the student's later work it will be important to know how e^{-x^2} varies when x is given different numerical values. Before substituting any particular number for x it will be advisable to proceed as follows: Let the function e^{-x^2} be denoted by y; then

$$y = e^{-x^2};$$

taking logarithms of each side this becomes

$$\log y = -x^2 \log e,$$

from which it follows that

$$-\log y = x^2 \log e.$$

Taking logarithms of each side again

$$\log(-\log y) = \log(x^2) + \log(\log e)$$

or

$$\log(-\log y) = \log(x^2) + \bar{1}.6378.$$

In the last equation it is easy to find the value of y that corresponds to any given value of x. The procedure is first to calculate x^2 and find its logarithm; then add $\bar{1}.6378$. The result is stated by the equation to be the logarithm of an expression enclosed in parenthesis, so the numerical value of that expression is easily found by the process of obtaining an antilogarithm. Then, when the

value of $(-\log y)$ has been obtained it is easy to write the value of $(+\log y)$, and from the value of $\log y$ there is no difficulty in finding y itself.

When such a determination is to be made for several different values of x it is convenient to arrange the various calculated quantities in the form of a table. Using the last of the equations that was derived from $y = e^{-x^2}$, and the tables of squares and of logarithms in the appendix, find in succession the values of x^2, $\log x^2$, $\log x^2 + \bar{1}.6378$, $\log(-\log y)$, $-\log y$, $\log y$, and y, for each one of the following values of x: 0, .2, .4, .6, .8, 1.0, 1.2, ... 2.8, 3.0; 4.0; 5.0. On the next unused left-hand page of your notebook tabulate the results in columns headed x, x^2, $\log x^2$, etc., carrying each line clear across the table, as shown in the illustration, before beginning the next line.

x	x^2	$\log x^2$	$\log (x^2) + \bar{1}.6378$	$\log(-\log y)$	$-\log y$	$+\log y$	y
0	0	$-\infty$	$-\infty$	$-\infty$	0	0	1
.2	.04	$\bar{2}.6021$	$\bar{2}.2399$	$\bar{2}.2399$	.0174	$\bar{1}.9826$	.961
.4	.16	$\bar{1}.2041$	$\bar{2}.8419$	$\bar{2}.8419$	.0695	$\bar{1}.9305$	.852
.6	.36	$\bar{1}.5563$	$\bar{1}.1941$	$\bar{1}.1941$			.699
.8	.64						
.							
.							

TABLE CONSTRUCTED WHEN FINDING THE VALUES OF e^{-x^2}.—It is important that the table be filled out line by line, not column by column.

Leave the opposite right-hand page of the notebook vacant until directions for using it are reached in another chapter.

68. Questions and Exercises.—1. Find the numerical values of $(1+\frac{1}{10})^{10}$; $(1+\frac{1}{100})^{100}$; and $(1+\frac{1}{1000})^{1000}$. Ans.: 2.6; 2.7; 2.7.

2. If log 10001 = 4.00004343 find the value of

$$(1 + \tfrac{1}{10000})^{10000}.$$

Ans.: 2.718.

3. Find the natural logarithm of π. Ans.: By definition, $e^x = \pi$ (§ 62). Solving this equation will give $x = \log \pi/\log e = 1.1447$.

4. Use the algebraical fact that $10^x 10^y = 10^{x+y}$ to prove equation 1 of § 63, which was merely stated without proof.

5. Turn back to the numbers that you have written in standard form (§ 60, ¶ 2) and write the characteristic of the logarithm of each of them. What relationship can be observed?

6. Prove that the second equation of § 67 must be true if the first one is true.

7. Write down any number that has three significant figures. Find its logarithm from the table, subtract it mentally from 0 (= log 1), and find the antilogarithm in order to obtain the reciprocal of the number that was written down. The mantissa is all that is needed for each logarithm, as the answer can be pointed off by inspection. For example, suppose the reciprocal of π is required: log 3.14 = 497; working from left to right, subtract each figure of the logarithm from 9 except the last one, which is to be subtracted from 10; result, 503; antilog 503 = 318; pointed off, 1/3.14 = .318.

Find 1 ÷ 269 by using the logarithm table, but without writing down any figures. The answer must be approximately .04.

Find the reciprocal of a four-figure number (for example, π or e) in the same way.

VI. SMALL MAGNITUDES

Apparatus.—Platform balance; set of gram weights; set of avoirdupois weights.

69. Approximate Values.—If the sum of one hundred dollars is put out at 3 percent compound interest it is said to " amount " to 103 dollars after one year, about 106 dollars after two years, about 109 dollars after three years, etc. The exact values of the latter amounts are 106.09 (the square of 1.03; in hundreds of dollars) and 109.2781 (the cube of 1.03); but if the principal had been one single dollar the amount that could have been repaid after two years or three years would necessarily have been \$1.06 or \$1.09, respectively, on account of the true amounts being less than \$1.06½ in the first case, and nearer to \$1.09 than to \$1.10 in the second. The fact that in United States money fractions of a cent cannot be used (except for purposes of calculation) corresponds exactly to the metrological fact that any measurement can be carried out to some particular degree of precision but no further.

70. Negligible Magnitudes.—Suppose a ruler graduated in centimetres and millimetres is used to measure the side of a square, and by estimating tenths of a millimetre the length is found to be 2.87 cm. The mathematical square of this quantity is 8.2369 cm², but it has already been seen (§ 58) that only three figures of this area can be trusted, because nothing is known about the fourth figure of the measurement from which it is derived. Accordingly, the square is said to have an area of 8.24 cm². Similarly, if one side of a square measures 1.03 cm., the measurement being correct to tenths of a

millimetre but nothing being known about hundredths of a millimetre, then its area will be correctly expressed by the quantity 1.06 cm^3, and the volume of a cube that has this square for one of its sides will be 1.09 cm^3. It should be noticed (*a*) that with an ordinary ruler it is impossible to measure a length of a few centimetres with an accuracy greater than is expressed by three significant figures; (*b*) that the area or volume calculated from such data cannot be trusted further than its third significant figure; (*c*) that the example just given suggests a remarkably simplified process of calculation where some quantity which is to be squared or cubed is *a little greater than unity*, namely: $(one + small\ amount)^2 = one + twice\ small\ amt.$, and $(one + small)^3 = 1 + 3(small)$.

Decide mentally, by induction, the value of $(1.-.02)^2$; then prove the result in two ways: (*a*) expanding in accordance with the binomial theorem; (*b*) squaring .98 by abridged multiplication.

The justification of the simplified process of raising a number which is approximately unity to a power will perhaps be made more evident by the following example:

Suppose that a metal cube has been constructed accurately enough to measure 1.00000 cm. along each edge. If it should be brought from a cold room into a warm room a delicate measuring instrument might show that the change of temperature had increased each dimension to 1.00012 cm. and by unabridged multiplication it would be easy to prove that the area of each side was 1.0002400144 cm^2 and that the volume had become 1.000360043201728 cm^3. If the most careful measurements make it just possible to distinguish units in the fifth decimal place then tenths of those units (represented by the sixth decimal place) would be impossible to measure, and the attempt to state not only tenths,

but hundredths and thousandths of those units would be absurd. By noticing that the number 1.0002400144 differs from the value obtained by abridged multiplication (1.00024) by only a few thousandths of the smallest measurable amount we can see clearly why the area of a 1.00012-cm. square is and must be 1.00024 cm^2. Similarly, the volume of the cube is neither more nor less than 1.00036 cm^3, and the string of figures running out ten decimal places further is absolutely meaningless.

It will be noticed that the number 1.0002400144 is in the same form as $1 + 2x + x^2$, the square of $(1 + x)$, where $x = .00012$; also that 1.000360043201728 corresponds to the cube, $1 + 3x + 3x^2 + x^3$. This is an illustration of the fact that when dealing with objects of the real world which is evident to our senses it may happen that a measured amount is so small that its higher powers, algebraically speaking, are relatively minute beyond all perceptive ability. Of course this must not be understood as meaning that the cube of a measurable length can ever be an impalpable volume; the cube of $1 + x$ is even a larger number, $1 + 3x$; it is the *difference* in size between this "physical" value, $(1 + x)^3 = 1 + 3x$, and the true mathematical value, $(1 + x)^3 = 1 + 3x + 3x^2 + x^3$, that eludes perception on account of x^2 being extremely small *in comparison with* x, which is itself minute.

71. Formula for Powers.—The examples that have been given above suggest that, if x is small enough, $(1 + x)^2 = 1 + 2x$, $(1 + x)^3 = 1 + 3x$, and in general $(1 + x)^n = 1 + nx$. The matter can be tested by making use of the binomial theorem:

$$(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)}{1 \cdot 2} x^2 \pm \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x^3 + \cdots.$$

This shows that $(1 \pm x)^n$ is equal to $1 \pm nx$ if x is so small that x^2, x^3, etc., are negligible, the only possible

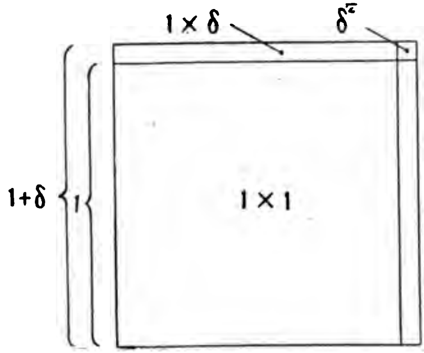


FIG. 13. THE SQUARE OF $1 + \delta$.—The fact that δ is small means that δ^2 must be very small. The error produced by taking the area $1 + 2\delta$ for the square of $1 + \delta$ is only a minute fraction of the total area.

exception being in case n should be so large that it could counterbalance the small size of x and prevent the term

$$\frac{n(n-1)}{1 \cdot 2} x$$

from becoming negligible. In the practical use of the formula, however, n is rarely larger than 2 or 3 while x is at most only a few hundredths and is usually very much smaller.

72. Properties of Deltas.—The small quantities which have been considered in this chapter are usually symbolized by the Greek letter δ. It has been seen that an important property of deltas is given by the equation $(1 \pm \delta)^n = 1 \pm n\delta$, but the student is advised to learn this in the form

$$(1 + \delta)^n = 1 + n\delta, \tag{1}$$

with the understanding that δ may have either a positive or a negative value; for example the square of $1+(-.03)$ is equal to $1 + 2(-.03)$ or $1 - .06$, or 0.94.

Find the value of each of the following expressions mentally: $.99^2$; $.98^2$; $.98^3$; $.97^2$; 1.00012^2; 1.00012^3; $(1 - .008)^2$; $.992^3$.

The ordinary process of algebraical division shows that $1/(1 + x) = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$. From this it follows that

$$\frac{1}{1+\delta} = 1 - \delta. \tag{2}$$

Divide 1 by 0.997. Ans.: $.997 = 1. - .003$; $1/(1-.003) = 1 + .003 = 1.003$.

Find mentally the reciprocal of 1.00012.

Find $1/(1.00012)^2$ mentally by using first formula (1) to simplify the denominator and then formula (2) to clear of fractions.

Find $(1.00012)^{\frac{1}{2}}$ and complete the following formula for yourself: $\sqrt{1 + \delta} = \cdots$.

If a number is so small that its square is negligible it will similarly be true that the product of two such numbers will be negligible. For example, if a number that is carried out to thousandth's place is 1.007 its square will not differ from 1.014 by a single thousandth: $1.007 \times 1.007 = 1.014049$; likewise 1.007×1.006 will not differ by a thousandth from 1.013, its "exact" value being 1.013042, as the student can easily prove by considering it to be a special case of $(1 + x)(1 + y) = 1 + x + y + xy$. Accordingly,

$$(1 + \delta_1)(1 + \delta_2) = 1 + \delta_1 + \delta_2. \tag{3}$$

This equation shows the advantage of keeping the signs positive and allowing the deltas to be either positive or

negative. If the deltas were restricted to positive values there would be a liability to error unless the equation could be written $(1 \pm \delta_1)(1 \pm \delta_2) = 1 \pm \delta_1 \pm \delta_2$.

Find mentally 1.04×0.98. Ans.: $.98 = 1. - .02$; $(1 + .04)(1 - .02) = 1.02$.

Find mentally 1.03×0.98; also $1.00012 \times .99890$ and prove the latter by abridged multiplication.

In your notebook work out neatly the value of 1.0021×1.0037 in three different ways: (*a*) by unabridged multiplication; (*b*) by abridged multiplication; (*c*) by the use of deltas, writing the two numbers one under the other and adding merely the deltas on paper in the customary manner. Compare the three processes carefully and draw the moral for yourself.

Remembering that a/b is the same as $a \times (1/b)$ write the formula for $(1 + \delta_1)/(1 + \delta_2)$.

A fact that is useful when making measurements of mass is *if two quantities are nearly equal to each other the square root of their product* (*or geometrical mean*) *can be obtained by taking their average* (*or arithmetical mean*). If the two quantities are denoted by a and $a + \delta$ (to indicate that their difference is a small magnitude) this statement can be proved as follows:

$$\left.\begin{aligned} \sqrt{a(a+\delta)} &= \sqrt{a \cdot a(1 + \delta/a)} \\ &= a\sqrt{1 + \delta/a} = a(1 + \delta/2a), \\ &\text{since } \delta/a \text{ is also a small magnitude. But} \\ a(1 + \delta/2a) &= a(2a + \delta)/2a \\ &= (2a + \delta)/2 = \frac{a + (a + \delta)}{2}. \end{aligned}\right\} \quad (4)$$

If an object appears to weigh m_1 when placed on one of the scale pans of a balance, and m_2 when on the other

pan it can be proved (see Fig. 14) that its true mass is $\sqrt{m_1 m_2}$. For example, a metal block weighed 15.19 and 15.23 gm. on the two sides of a balance. Its true weight is accordingly $\sqrt{15.23 \times 15.19}$ gm. Equation (4) shows that this is the same as 15.21 gm.

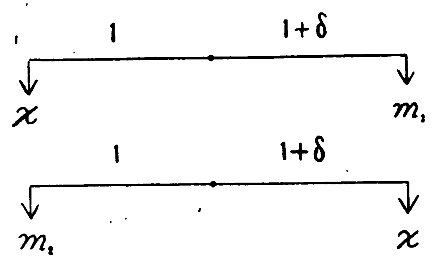


FIG. 14. DIAGRAM OF A BALANCE.—The force multiplied by the distance at which it acts is the same on each side $\therefore x = m_1(1+\delta)$; also $m_2 = x(1+\delta)$. Eliminating $(1+\delta)$ gives $x/m_1 = m_2/x$, whence $x = \sqrt{m_1 m_2}$.

Weigh your whole set (16 ounces) of avoirdupois weights, considered as an unknown mass on the platform balance against the brass gram weights. Repeat the process on the other pan of the balance, remembering that the reading of the sliding weight is additive in one case and subtractive in the other. Find the true weight, in grams, of the avoirdupois set.

It will be almost self-evident from Fig. 15 that

$$\frac{\tan \delta}{\delta} = 1 \quad \text{and} \quad \frac{\sin \delta}{\delta} = 1 \tag{5}$$

if it is remembered that an angle is measured by the ratio of arc to radius.

By consulting the table of circular functions find the largest whole number of degrees for which $\tan \delta = \delta$ to four decimal places. The numerical measure of

each angle will be found in line with the number of degrees, but in the column headed RAD, an abbreviation of *radian measure*, which is another synonym for circular measure.

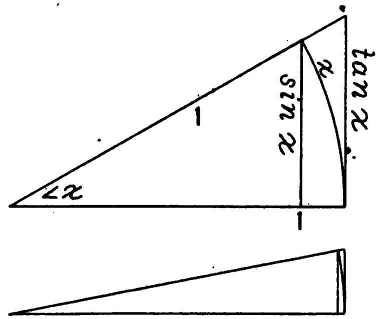


FIG. 15. FUNCTIONS OF A SMALL ANGLE.—The tangent is always larger than the angle and the sine is always smaller, but the ratio of either function to the angle comes nearer and nearer to unity as the angle is made smaller and smaller. A formal proof of the fact that lim sin x/x = lim x = lim tan x/x can easily be worked out if two equal angles are juxtaposed so that their sine lines complete the chord of the double arc.

If an accuracy of three decimal places is all that is needed how large can δ be without differing from tan δ? Ans.: 6°; but not 7°. How large if δ is to equal sin δ?

73. Transformation of Operands.—It will be noticed that most of the rules that have been given for operating upon small quantities involve functions of $1 + \delta$, not of δ alone. This means that factoring, or some other process, is often necessary in order to put an expression into the form of $1 + \delta$. For example, $\sqrt{50}$ is equal to $\sqrt{7^2 + 1}$, an expression which is not in the form of a function of $1 + \delta$ but can be made so by dividing the binomial by an assumed factor 49: $\sqrt{49 + 1} = \sqrt{49(1 + 1/49)} = 7\sqrt{1 + 1/49} = 7(1 + 1/98)$. Now

it will be noticed that 98 is a little less than 100, and so

$$7\left(1+\frac{1}{98}\right)=7\left(1+\frac{1}{100-2}\right)=7\left(1+\frac{1}{100(1.-.02)}\right)$$

$$=7\left(1+\frac{1}{100}(1+.02)\right)=7(1.0102)=7.0714.$$

The first four figures of this result are correct. Extreme accuracy cannot be expected where a delta is as large as one or two percent; in most physical calculations it is much smaller than this.

74. Recapitulation.—The formulæ for calculation with small quantities are collected here for reference. Notice that the second and fifth are special cases of the first, and the first formula, *for n equal to a whole number only,* is a special case of the fourth,

$$(1+\delta)^n = 1+n\delta,$$

$$1/(1+\delta) = 1-\delta,$$

$$\frac{1+\delta_1}{1+\delta_2} = 1+\delta_1-\delta_2,$$

$$(1+\delta_1)(1+\delta_2)(1+\delta_3)\cdots = 1+\delta_1+\delta_2+\delta_3\cdots,$$

$$\sqrt{1+\delta} = 1+\tfrac{1}{2}\delta,$$

$$\sqrt{a(a+\delta)} = \frac{a+(a+\delta)}{2} = a+\delta/2,$$

$$\frac{\tan\delta}{\delta} = \frac{\sin\delta}{\delta} = 1.$$

75. Questions and Exercises.—1. Find the value of $\frac{400}{797}$. (Suggestion: divide both numerator and denominator by 800 in order to convert the latter into $1-\delta$.)

2. Find the value of $.504 \times .498$. (Suggestion: the first of these is a little more than 1/2, *i. e.*, is equal to 1/2 multiplied by a little more than one; likewise, the second is a little less than 1/2.)

3. Write the general formula for the value of the expression $(1 + \delta_1)^m(1 + \delta_2)^n(1 + \delta_3)^p(1 + \delta_4)^q \cdots$. How many of the formulæ that are given in § 74 does it include?

4. When warmed up to ordinary room temperature the brass scale of a barometer is too long; as a result its reading falls short of the correct number of centimetres and needs to be multiplied by 1.00037. The mercury, however, expands at a different rate from the brass and tends to make the reading too great; to correct this error alone the reading must be divided by 1.00364. The correct height of the mercury will accordingly be obtained if the observed height is multiplied by 1.00037/1.00364. What is the percentage of difference between the corrected reading and the original reading? (Suggestion: call the observed height unity.)

5. In the equations of Fig. 14 eliminate x instead of δ and show that $1 + \delta = \sqrt{m_2/m_1}$. Then suppose that $m_2 = m_1 + \Delta$ and show that $1 + \delta = 1 + \Delta/2m$, and hence that $2\delta = \Delta/m$. If the two weighings of your avoirdupois pound (§ 71) differed by 0.3 percent how much do the two arms of the balance differ in length?

6. If $\sqrt{10} = 3.16$ find the value of π^2. (Suggestion: $\pi = 3.14 = 3.16(1 - 2/316)$).

VII. THE SLIDE RULE

Apparatus.—A slide rule provided with A, B, C, D, L, S, and T scales, a runner, and a list of equivalent measures.

76. Addition by Means of Two Scales.—If two scales of centimetres should be laid parallel so that the zero of the second one coincided with the seventh division of the first (Fig. 16) it would be evident that the fifth division

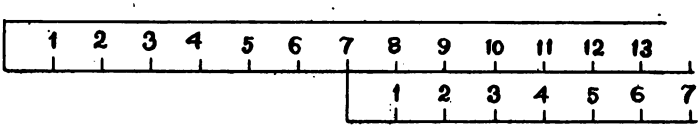


FIG. 16. ADDITION WITH UNIFORM SCALES.—By starting afresh at 7 of the original scale and measuring 5 (or any other number, n) on the new scale we reach the distance of $7 + 5$ (or $7 + n$, as the case may be) on the original scale. It should be noticed that this illustrates subtraction as well as addition: $12 - 5 = 7$, also the difference between any other pair of opposite scale numbers is equal to 7.

of the second would be opposite the twelfth division of the first. In other words, when a length of five has been added in this manner to a length of seven the result can be seen at a glance to be equal to twelve. Notice also that the arithmetical difference between scale divisions that lie opposite each other is everywhere the same, and that it is equal to the number on the first scale which is opposite the beginning of the second.

77. Multiplication with Logarithmic Scales.—If the same experiment is tried with two scales whose divisions are not a succession of whole numbers but are the logarithms of such a series the result will be different,

for adding logarithms corresponds to multiplying natural numbers. Accordingly if we start at log 7 (Fig. 17) and

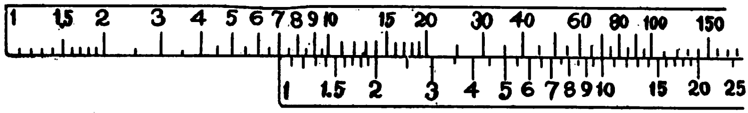
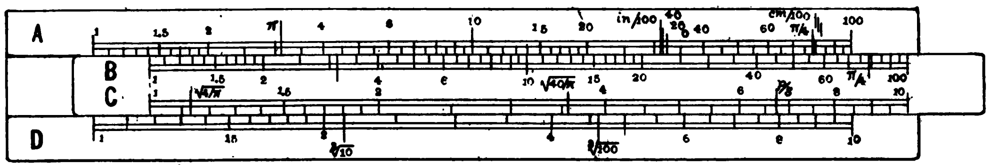


FIG. 17. MULTIPLICATION WITH LOGARITHMIC SCALES.—The distances on each scale which are marked 1, 2, 3, etc., are really equal to log 1, log 2, log 3, etc. Since addition of logarithms effects multiplication of their natural numbers it will be seen that the number above the 5 of the lower scale is not 7 + 5 as in Fig. 16, but is equal to 7 × 5, or 35. Since subtraction of logarithms corresponds to division it will be evident that the opposite pairs of numbers have their quotients equal to seven, instead of their differences.

measure a further distance of log 5 we shall come out with log 7 + log 5, which is not equal to log (7 + 5) but to log (7 × 5). Notice not only that 5 on the second scale comes opposite 35 on the first, but also that quotients of corresponding numbers are everywhere equal to 7, just as differences are in Fig. 16; and furthermore, the upper scale in Fig. 17 forms a multiplication table (a seven table in this case) for the numbers on the lower scale just as it did an addition table in Fig. 16.

78. The Slide Rule.—The apparatus called a *slide rule* is essentially a ruler containing a groove in which is a movable slide. Logarithmic scales are so marked that one of them (the "*A* scale") can be held stationary while another (the "*B* scale") can be placed in any required position below it (Fig. 18).

Two scales (*C* and *D*) along the lower edge of the slide can be used in the same way and can be read a little more accurately on account of their subdivisions being larger, but the two upper scales (*A* and *B*) will be found the most convenient for general use. Each one of them is really



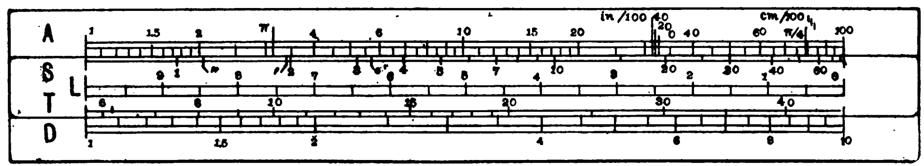


FIG. 18. THE SLIDE RULE.—The essentials of the slide rule are four logarithmic scales arranged in two pairs and called A, B, C, and D, as shown in the diagrams even if not lettered on the apparatus.

FIG. 19. THE SLIDE RULE, SHOWING THE REVERSE SIDE OF THE SLIDE.—Opposite any angle on S is found its sine (on A); opposite any angle on T is found its tangent (on D). The numbers on L are the logarithms of those on D, but arranged backward so that they can be used when the A, B, C, and D scales are in their usual position.

two complete logarithmic scales, as will readily be seen, but its right-hand half is often marked 10 to 100 instead of 1 to 10. In calculating with the aid of the slide rule it is advisable to ignore all decimal points while using the apparatus, and to point off the required answer by making a rough preliminary calculation mentally, as illustrated in § 14. It is then possible to use either the 7, for example, in the left-hand part of the *A* scale, or the 70 (or 7) in the right-hand part, whichever may happen to be the more convenient, in order to represent 7000, or 700, or .07, or any other number whose significant figure is 7.

By consulting Fig. 17 find the value of 70×20. Ans.: 1400. Find $3\frac{1}{2} \div 5$. Ans.: 0.7, because it is evident that the quotient must be somewhat less than 1. Find the approximate value of $.0007 \times 1.4$. Ans.: nearly .0010. Find 70×45. Ans.: as nearly as the value can be read from the diagram, it seems to be 32; since $70 \times 40 = 2800$ the required product must be 3200, the ciphers not being significant. Two-figure accuracy is the highest that can be obtained from Fig. 17.

79. Reading a Logarithmic Scale.—The ordinary slide rule is usually constructed so carefully that three-figure accuracy is always attainable, although it requires the "estimation" (§ 46) of halves or fifths or even tenths of the graduated intervals. This is sufficient for the great majority of practical calculations in every-day life; if four-figure accuracy or greater is required it is usually more convenient to use logarithm tables, but a slide-rule makes a 3-figure logarithm table superfluous because it can be used with greater rapidity.

It will be observed, both in Fig. 18 and on the slide rule itself, that the distances from log 1 to log 2, log 2 to log 3, log 3 to log 4, etc., grow progressively smaller and smaller, so that those near the left-hand end of the scale

can be more finely subdivided than those toward the right-hand end. Thus it should be noticed that on the *A* or *B* scale of the ordinary slide rule the third subdivision to the right of 7 or 70 is 73, while on the *C* or *D* scale it is 715; but the third subdivision to the right of 1 represents 106 on the upper scales and 103 on the lower ones. To avoid mistakes in picking out the position of a given number it is best to find both the next larger single figure and the next smaller one; then look for the long .5 graduation which is about half-way between them.

Find 365 on the *A* scale of your slide rule by locating first 3, then 4, then 3.5, then 3.6 and 3.7, and finally 365. Find 235 in the same way; then by estimation set the transparent runner with its vertical line as nearly as possible at 238. Move the slide so that the end (1) of the *B* scale comes under this 238, and see if 210 comes just under 500. Set the transparent runner at 128; then at 106, and 109. Repeat the work of this paragraph using the *D* and *C* scales *respectively* instead of the *A* and *B* scales.

In reading a given position on the scale it is likewise best to read the next larger single figure and the next smaller one, also the half-way point between them.

Set the left-hand 1 of the *B* scale at random somewhere between 1 and 2 of the *A* scale. Then read its position accurately; also the positions of 2, 3, 4, 5, 6, 7, 8, and 9. On some slide rules the subdivisions between 1 and 2 of the *C* and *D* scales are marked 1, 2, 3, $\cdots$ instead of 1.1, 1.2, 1.3, etc. If this is the case with your slide rule it will be necessary, when looking for a number such as 7, to see that it belongs in the series that runs 7, 8, 9, 10, out to the right-hand end of the rule, and not in the series that runs merely from 1.0 to 2.0.

80. Multiplication.—To find the product of two numbers with the slide rule either end of the B scale should be set opposite one factor on the A scale, then the other factor on the B scale will be opposite the required product on the A scale. *Try this process with small numbers* by setting 1 on B opposite 3 on A and observing that the position of 2 on B gives the product 6 on A.

Multiply 2×4; 2×5; 2×6; 3×9; 7×8. Find 7×13, and 7×16, remembering that the two halves of the B scale (and also those of the A scale) are identical, so that if 16 (or 1.6) when taken on the right half of the B scale falls beyond the end of the A scale the same result can be read over 16 (or 1.6) on the left half.

Multiply 1.5 by 2; 1.8×2; 1.7×2.1 (estimate the third figure of the product); 17.8 by 2; 1.79 by 2.53.

Find 0.6×183 (caution: not 1.6×183; see end of § 79); 7.3×1.09; $325. \times 106.5$; 0.073×0.0016. If you have any difficulty in pointing off the last product master it before going any further. The use of "standard form" often makes the preliminary checking easier; thus in the last two examples given 3×10^2 multiplied by 1×10^2 gives 3×10^4 or 3000; and $(7 +) \times 10^{-2}$ multiplied by $(2 -) \times 10^{-3}$ gives 14×10^{-5}, or .00014.

81. Division.—In Fig. 16, above, notice that a length of twelve may be considered as being laid off from left to right, and then, beginning at the point 12, a length of 5 is laid off to the left, or subtracted from the original 12, giving a final result of 7. Similarly, in fig. 17, log 35 minus log 5 equals log 7. The rule for division is accordingly: place the divisor on B under the dividend on A and read the answer on A over either end (or the middle) of the B scale. Such directions should never be memorized by the student, but he should practice the process until he becomes familiar with it, and in case of

doubt should *experiment with small numbers* where the answer is obvious beforehand.

Divide 35 by 5; 30 by 5; 30 by 4; 11 by 4; 11 by 7; 11 by 10; 11 by 11; 11 by 12; 11.8 by 99.; 114. by 3.4; 3.4 by 114.

82. Ratio and Proportion.—In Fig. 16 it will be seen that $8 - 1 = 9 - 2 = 10 - 3 = 7$. Since subtraction of logarithms corresponds to division of natural numbers Fig. 17 shows that $14 \div 2 = 21/3 = 28 : 4 = 35/5$, etc. That is, with the slide set in a given position any two opposite numbers are in the same ratio as any other two. *Test this on the slide rule with small numbers;* for example, set 6 under 2 and notice that 15 is under 5, for $6 : 2 :: 15 : 5$. Solve the following proportions by setting the rule so that the answer is always found on the A scale.*

$$6 : 2 :: 15 : x,$$
$$2 : 6 :: 15 : x,$$
$$3 : 2 :: 9 : x :: 12 : y :: 10 : z.$$

Solve $31 : 750 :: .005 : x$; first notice that 750 is about twenty times as large as 31. Solve $2300 : .036 :: 990 : x$. If 26 inches = 66 centimetres solve the following equation as accurately as possible: $26 : 66 :: 1 : x$. What is the length in centimetres of 1 inch? From the slide rule find the *approximate* length of 4 inches. How many centimetres in 41 inches?

Set 1 precisely under the special mark that indicates π (= 3.142) and notice that 7 is nearly under 22 because 22/7 is approximately equal to π. Look along the scales for another ratio which is equal to π, and find one that is not numerically equal to 22/7 or to 3.142857. . . .

* To avoid uncertainty about which scale should be read it is advisable to get into the habit of setting the slide rule so that the answer will always be found on one of the fixed scales, A or D; not on either of the movable ones, B or C.

Then find its decimal value by *unabridged* division carrying out the work until the figures become different from those of the correct value of π, as was done in § 49.

83. Equivalent Measures.—Turn to the back of the slide rule and find a statement about centimetres and inches. It will usually be given in tabular form, such as cm: in. . . . 26: 66, together with various other data; if not, use the table given here. Find a statement of the

26 in = 66 cm	30 atmo = 31k/cm^2
292 ft = 89 m	128 lb/in^2 = 9 k/cm^2
35 yd = 32 m	500 lb/in^2 = 34 atmo
87 mi = 140 km	
	340 ft-lb = 47 kgm-m
31 in^2 = 200 cm^2	134 hph = 100 kwh
140 ft^2 = 13 m^2	67 kwh = 58000 Cal
990 ♏ = 61 cm^3	$\pi = 355/113$
23 f℥ = 680 cm^3	$\sqrt{\pi} = 39/22$
36 in^3 = 590 cm^3	
14 gal = 53000 cm^3	$\angle 1 = 57^\circ\, 17'\, 45''$
	$= 3437'.7$
108 gr = 7 gm	$= 206264''.8$
9 ℥ = 280 gm	$1^\circ = 1.74533 \times 10^{-2}$
1940 av oz = 55 kgm	$1' = 2.90888 \times 10^{-4}$
22 lb = 10 kgm	$1'' = 4.84814 \times 10^{-6}$

Table of Equivalents for Use with the Slide Rule.—The particular numbers are chosen so as to be accurate as far as three significant figures, at least. Thus not only does 26 in. equal 66 cm., but also 26.0 in. = 66.0 cm. The value of π is not only 355/133, but also 3550/1330, and 35500/13300, even to 355000/133000, as may easily be verified by the process of abridged division.

relation between pounds and kilograms and calculate your own weight in kilograms, remembering to set the slide rule so that the answer will be found on the *A* scale.

84. Reciprocals.—Set 4 or 3 or any other small

number on B under 1 on A, and notice that above 1 (or 10 or 100) of the B scale will be found .250 or .333 or the reciprocal of whatever small number was used. Verify $1 \div 7 = .142857$, as accurately as the apparatus will allow.

85. C and D Scales.—*Experiment with any small number*, using the C and D scales, and show where it must be set in order to find its reciprocal on the (fixed) D scale. Notice that if 8 on C is set over 10 on D the value of 1/8 will be found under 1, but if 2 is set over 1 the value of 1/2 will be found under the 10 instead.

Try multiplication, division, proportion, and equivalents on the C and D scales, remembering to set the slide so that the answer always comes on the stationary (D) scale. If the method for any of these processes has been forgotten *experiment with small numbers* so that you know the required answer beforehand. In multiplication if the answer runs off the end of the rule set 10 instead of 1 on C opposite the first factor, and in division read the result under 10 instead of under 1 when necessary. If the fourth term of a proportion cannot be read the runner must be placed over the end of the C scale and the slide then moved so that the other end of the C scale is under the runner. As an example, set 26 on C over 66 on D and show that the number of centimetres in 5 inches is 12.7.

86. Squares and Square Roots.—Set the slide so that the ends of the B and C scales coincide with the ends of the A and D scales. Then move the runner so that its vertical line falls on 9 of the C and D scales and notice that it also comes on 9^2 or 81 of the A and B scales. Set it at 8, 7, 6, 5, etc., on the lower scales and notice that the number on the upper scales just over n on the lower ones is always n^2. For a slide rule that has no runner set

1 on C over n on D and 1 on B will indicate n^2 on A. Read the square of 12; of 13; of 19. Find the (four-figure) square of 43, knowing that the last figure must be 9.

Under any number on the A (or B) scale will be found its square root on the D (or C) scale, but it must be remembered that any given arrangement of significant figures has two different square roots, according to how it is pointed off (compare § 13, ex. 41). For example, $\sqrt{1500} = 40 -$; $\sqrt{150} = 12 +$; $\sqrt{15} = 4 -$; $\sqrt{1.5} = 1.2 +$; $\sqrt{.15} = .4 -$; $\sqrt{.015} = .12 +$; etc. Notice that under 150 of the left-hand half of the A scale one of these roots, 122, is found; while the other root, 387, occurs under the 150 of the right-hand half. Since one of these is about three times as large as the other the simple precaution of making mentally a rough preliminary calculation of the root will avoid the possibility of obtaining the wrong number. For example, is the square root of .036 given by the significant figures 19 or 60, and how should they be pointed off?

87. Compound Operations.—A quotient such as m/n is found by setting n on the B scale under m on the A scale and reading the value opposite the end of B. (If this is not perfectly obvious *try it with small numbers.*) If m/n is further to be multiplied by some other number, say x, notice that the slide rule does not need to be re-set as it is already arranged so that the required product will be found over x of the B scale. Find two sevenths of thirteen in this way and notice particularly that it is not necessary to read the value of the fraction 2/7; merely set 7 under 2 and find the required answer over 13.

Similarly, m^2n will be found on A opposite n of the B scale if 1 on C is set at m on D; and $\sqrt{mn}$ will be found on D under n on B if 1 on B is set to m on A.

A series of fractions like a/m, b/m, c/m, d/m, $\cdots$, can be read off merely by setting the slide rule for $1/m$ and looking opposite a, b, c, d, $\cdots$.

The slide rule is usually made with two "cylinder points" marked on the C scale at $\sqrt{4/\pi}$ and $\sqrt{40/\pi}$. By placing either one of these opposite the diameter of a cylinder the length of the cylinder on B will be found to indicate its *volume* on A.

There is usually a special mark for π on the left-hand half of the A and B scales, and for $\pi/4$ (or .7854) on the right-hand half. By placing the end of the B scale opposite the latter the area of any circle will be found on A opposite the diameter on D.

88. Determination of Circular Functions.—In most slide rules the back of the sliding part is provided with three scales, which are named, and sometimes marked, S, L, and T, from above downward. If the slide is placed so that the ends of the S and T scales coincide with the ends of the A and D scales respectively, the sine of any angle from 1° to 90° will be found on A opposite the number of degree and minutes on S; and the tangent of any angle from 6° to 45° will be found on D opposite the number of degrees and minutes on T. The decimal point is located by recalling the facts that $\sin 90° = 1$, $\tan 45° = 1$, and if two angles differ only slightly their sines (or tangents) will also be only slightly different.

By using the slide rule show that $\sin 70° = .940$; write the values of $\sin 50°$; $\sin 30°$; $\tan 30°$; $\sin 11° 30'$; $\tan 11° 30'$; $\sin 6°$; $\sin 5°$; $\sin 1°$.

Since $\tan (45° + a)$ and $\tan (45° - a)$ are reciprocals of each other (prove by substituting $45 + a$ for x in the equation of § 45; 12, c), tangents of angles greater than 45° are easily obtained from the slide rule. For example, to find $\tan 49°$, which is $\tan (45° + 4°)$, set 41°, which is

$45° - 4°$, opposite 10 on D and read the required value, 1.15, on D opposite the left-hand of the T scale.

For sines less than .01 and tangents less than 0.1 different types of slide rule employ different methods, usually based on the formulæ $\sin \delta = \tan \delta = \delta$ (§ 72). If no special marks for angles are found on the C scale or the S scale use the equations for 1°, 1′, and 1″ in § 83.

89. Determination of Logarithms and Antilogarithms.—Set 1 on C opposite any number, n, on D, and $\log n$ will be found on L opposite a special mark on the *back* of the slide rule. *Try this with small numbers* whose logarithms are already known; *e. g.*, $\log 3 = 0.477$.

90. Questions and Exercises.—1. If decimal points are disregarded one square root of a given arrangement of significant figures is stated (§ 86) to be about three times as large as the other. What is the exact value? Why?

2. Prove that the volume of a cylinder, $\pi r^2 l$, is correctly obtained by the process given in § 87.

3. Explain how it is that the process in § 88 for finding tan 49° really gives the *reciprocal* of tan 41.

4. Set 1 on the C scale over a tentative cube root of n and see whether $\sqrt[3]{n}$ on B comes under n on A. Practice this method of finding cube roots. In what way would the marks $\sqrt[3]{10}$ and $\sqrt[3]{100}$ in Fig. 18 be helpful?

5. Find a way to use the marks "in/100, 40 20 0" opposite $32\pm$ of the A scale in Fig. 18 in order to reduce "American Wire Gauge" to actual diameters of the wires in hundredths of an inch. Start from the following data:

No. 0 = .325 inch, no. 1 = .289, no. 2 = .258, no. 3 = .229, no. 4 = .204; no. 8 = .128, no. 12 = .081, no. 16 = .051, no. 20 = .032, no. 24 = .020, no. 28 = .013, no. 30 = .010. Notice the marks "cm/100" that are located [log] 25.4 times as far to the right, and show that the diameter of no. 18 wire is very nearly 1.00 mm.

VIII. GRAPHIC REPRESENTATION

Apparatus.—A pencil with a *sharp* point for marking positions accurately on the squared paper of the notebook.

91. Indication of a Point by Two Numbers.—The position of any point on the surface of the earth may be located by two numbers. One of these, the longitude of the point, expresses its distance to the east or west of an arbitrary line, the meridian of reference. The other, its latitude, gives its distance north or south from a definite base line, the equator.

92. Representation of Two Numbers by a Point.—In almost all branches of science a process which is just the opposite of that given above has been found to be of very great value: instead of using two numbers to describe the location of a point the method is reversed and any two related numbers are represented diagrammatically by the position of a point. For example, in order to indicate that the out-door temperature on January eighth was 21° F., a point could be marked on a sheet of paper at a distance of 8 arbitrary units from the left-hand edge of the paper and at the same time 21 units above the bottom of the sheet. If the temperature had fallen to 17 on Jan. 9, another point could be marked, located 9 units from the left-hand edge and 17 units above the lower edge of the paper.

93. Representation of Two Variables by a Line.—By continuing the process of marking down, or "plotting" a new point for each successive day and its temperature a series of points would be obtained; and if the temperatures should be observed at more frequent intervals, every hour or every minute, the points would come so

close together that they would almost make a continuous line. The ups and downs of the irregular line would indicate clearly to the eye the fluctuations of temperature that correspond to the onward march of time, which would be indicated by the steady progress of the line from left to right. The objectionable feature of a diagram of this kind would be that no temperature below zero could be represented.

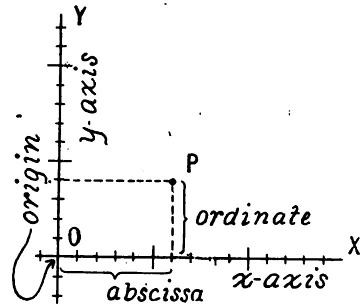


FIG. 20. GRAPHIC DIAGRAM.—The two numbers 6 and 4 are represented by a point (P) whose coordinates are the abscissa, or x-value, of 6 units measured horizontally, and the ordinate, or y-value, of 4 units measured vertically. Positive values are always measured to the right and upward; negative ones, to the left and downward.

94. Graphic Diagrams.—In order to allow the representation of negative values of a variable such as temperature it is customary not to measure from the bottom of the paper but from an arbitrary horizontal line ruled at a sufficient height to allow space for the data that are to be indicated. The other variable is not always time, but may be a changing quantity which also assumes negative values, so that a vertical line of reference must be ruled at some distance from the left-hand edge of the paper. A *graphic diagram* consists of these two lines of reference, called *axes*, a numerical scale along each of them, and the point or assemblage of points which corresponds to the numerical values that are to be represented. The vertical distance of any point from the horizontal line, or *x-axis*, is called the *ordinate* of that point (Fig. 20); and the horizontal distance from the

vertical line of reference, or y-axis, to this ordinate is called the *abscissa* of the point In the diagram the point P has an abscissa of 6 and its ordinate is 4. The abscissa of a point, being measured along the x-axis, is sometimes called the *x-value* of the point; and the ordinate, or vertical distance, is called the *y-value*. Considered collectively, the two distances are called the *coordinates* of the point. The point whose coordinates are both zero, *viz.*, the intersection of the two axes, is called the *origin* of the graphic diagram.

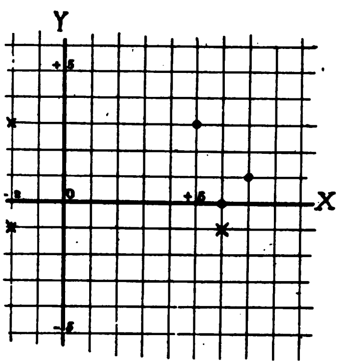


Fig. 21. Points on a Graphic Diagram.—The points shown by small dots have no negative values. The points shown by crosses have negative values for one or both of their coordinates.

95. Practice in Plotting Points.—Figure 21 shows three points represented by small dots and three others marked by crosses. The highest dot has an x-value of $+5$ and a y-value of $+3$; when it is considered alone it may be spoken of as "the point (5, 3)," the coordinates being written, x-value first, in parentheses and separated by a comma. Notice that the other dots are located at (6, 0) and (7, 1). One of the crosses is located at $(-2, +3)$. Write down the location of each of the other two.

Draw two short axes in your notebook. Mark their positive directions X and Y, as in Fig. 21. Lay off a short scale of positive or negative numbers along each axis, numbering every line, or every other line or every fifth

line, as you prefer. Make a dot for each of the following points: (1, 2); (2, 1); (3, 0); (4, − 1); (5, − 2). Without drawing new axes plot the following points with small ×'s on the same diagram: (4, 0.5); (0, − 1.5); (2, − 0.5); (− 1, − 2); (1, − 1); (3, 0). On the same diagram use small +'s for the following: (− 0.5, 0); (+ 0.5, − 2); (− 2, + 3); (0, − 1); (− 1.5, + 2); (− 1, + 1). Fractional values are most conveniently plotted by drawing short horizontal and vertical lines that intersect to make a +. On the same diagram plot the following points in this way and then connect them by a smooth free-hand curve:

$x=$	0.6	1.0	2.0	2.3	3.0	3.7	4.0	5.0	6.0	7.0	7.7
$y=$	3.0	3.1	3.1	3.0	2.5	2.0	1.6	1.0	0.5	0.6	1.0

Take one of the following tables, I–V, as indicated by your instructor, and locate all of its points by means of small +'s on a new graphic diagram. Before drawing the axes find the largest and (algebraically) smallest of the x-values in the table and see that the y-axis is not drawn too far to the right or left to leave room for them. Examine the y-values in the same way and draw the x-axis as closely under your previous notes as they will allow. Do not start the diagram on a new page or place it so as to use up an unnecessary amount of space. Do not draw lines to connect the separate points.

96. Orientation of a Graphic Curve.—On the squared paper of your notebook draw a rectangle that includes 128 of the small squares, making it 8 squares wide and 16 squares high. Examine the table on the next page (Table A) and locate the x-axis and y-axis so that none of the points representing the tabular values shall fall outside of the limits of the rectangle. Draw a second

x	y
0	1
−1	0
0	−1
1	0
2	1
3	2
4	3
3	4
2	3
2	−1
4	1
5	0
7	0
4	−3
3	−4
1	−7
5	−6
2	−3
3	−2
4	−1
2	−5
6	−1
6	1
5	−7
6	−7
0	−7
1	−6
3	−7
4	−5

TABLE I

x	y
−1	+1
1	4
−2	3
1	5
−1	0
−3	1
−7	0
−6	−1
−2	5
−5	2
−4	1
0	5
−4	−1
−5	4
−4	5
−2	−1
−7	−1
−5	3
−3	2
−1	3
−5	−1
−3	5
−1	2
−3	−1
−5	1
−1	−1
−3	3
−5	5
−1	5

TABLE II

x	y
0	2
1	1
0	−1
1	−2
−3	−2
−2	−3
0	−3
1	−4
4	−1
5	−2
1	−5
2	−6
2	−1
4	−3
−1	0
−2	−1
0	−6
1	−6
−1	−4
1	−3
2	2
1	2
3	−4
2	−3
1	0
3	0
1	−1

TABLE III

x	y
1	1
−1	0
2	3
−2	5
−1	3
0	1
1	0
−1	2
−2	1
−5	0
−4	−1
−2	−1
0	−3
1	−2
3	2
−3	3
1	−1
−3	2
1	−3
3	3
−3	4
−5	−1
0	3
−3	−1
−1	−1
−3	1
−1	1
1	3
−3	5

TABLE IV

x	y
1	1
2	0
−6	0
−8	−2
−6	−2
−8	0
2	−2
0	0
−3	0
0	−2
−3	−2
−7	−3
−3	3
−4	−1
−5	1
−3	−1
−2	−4
−5	−3
−1	1
3	−1
−1	−3
−7	1
−4	2
−2	2
−3	−5
1	−3
−4	−4
−9	−1
−2	−1

TABLE V

rectangle of the same size and plot the values of Table *B* in it without going beyond its boundaries. Examine Tables *C* and *D* and plot a graphic diagram for each one,

x	y
0	−5
1	−8
2	−9
3	−8
4	−5
5	0
6	7
−1	0
−2	7

TABLE *A*

x	y
0	−1
1	+1
2	3
3	5
4	7
5	9
−1	−3
−2	−5
−3	−7

TABLE *B*

x	y
0	−4
1	−5
2	−5.6
3	−6
5	−6.5
−1	−2
−1.5	0
−2	4

TABLE *C*

x	y
0	0
0.5	0.25
1	1
2	4
3	9
4	16
5	25
−1	1
−2	4
−3	9

TABLE *D*

in one of the rectangles and with the axes as they have already been drawn if possible, otherwise in a third rectangle which is to be of the same size as the previous ones but in which the scales of x-values and y-values may be *condensed*, so that the dimensions of each square may represent two, or five, or ten units, as may be most convenient, or may be *expanded*, so that one unit may be represented by two or more times the length of a single square.

In the next table notice that the y-values lie between the extreme values of $+ 42$ and $+ 48$. In such cases there is usually no objection to dispensing with the x-axis

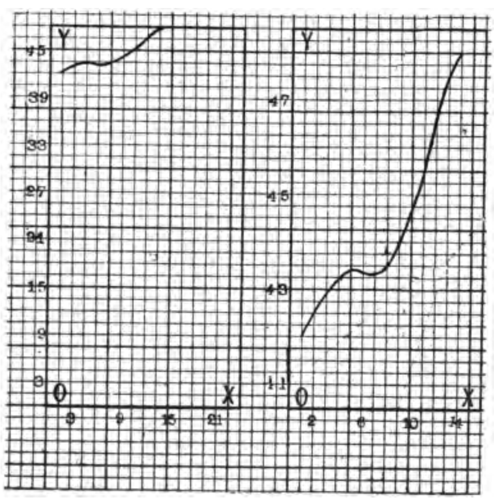


FIG. 22. GRAPHIC DIAGRAM WITH CONDENSED SCALES.—Notice that the x-values have been so condensed that the curve does not extend far to the right; and the y-values have been condensed to the same extent, making the curve relatively flat.

FIG. 23. GRAPHIC DIAGRAM.—Notice how both scales have been arranged so that the same table as was illustrated by Fig. 22 now has its values much better displayed.

on the diagram if the scales are plainly indicated. In Fig. 22 there is much wasted space between the curve and the x-axis, and the curve itself is relatively flat. Notice how both of these objections have been overcome in Fig. 23.

$x =$	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$y =$	42	$42\frac{1}{2}$	43	$43\frac{1}{4}$	$43\frac{1}{2}$	$43\frac{1}{3}$	$43\frac{1}{3}$	$43\frac{1}{2}$	44	$44\frac{3}{4}$	$45\frac{1}{2}$	$46\frac{1}{2}$	$47\frac{1}{2}$	48

TABLE OF VALUES REPRESENTED IN FIGS. 22 AND 23.

97. Choice of Scales.—When only a few points are to be plotted and the range of extreme values is not very great, as in tables A, B, C, and D of § 96 it is advisable to use a normal scale of one unit for each square of the ruled paper. All of the graphic diagrams in this chapter and the next one are to be constructed in this way unless special directions are given to the contrary. Condensed scales are most useful for large numerical values, including data given in round numbers, and for keeping a diagram within the limits of a definite space. Expanded scales are necessary for minute values and those which need accurate representation. For example, when the width of a square is only half a centimetre (one fifth of an inch) or less it is possible to divide it into tenths by estimation and to show a perceptible difference in position to correspond to a difference of one tenth in numerical values, but values differing only by a number of hundredths or thousandths need an enlarged scale to make their variations show in the graphic diagram. In some cases the importance of the relation or ratio of x-value to y-value makes it necessary that both x-scale and y-scale shall be the same, whether normal or condensed or extended. In other cases the two sets of values represent measurements which are mutually incommensurable (as in the case of time and temperature,

§§ 92 and 93), so that it is theoretically of no importance whether the two scales correspond or not. Where the slant of the curve, or of some important part of it, is fairly uniform, however, it is often more satisfactory to choose the scales in such a way as to make the slope approximately 45 degrees, as in Fig. 23.

98. General Principles of Plotting.—When one of two variable quantities changes as the result of changing the other it is customary to use the horizontal scale of x-values for the independent variable and the vertical scale of y-values for the dependent or resultant variable. Thus, for a graphic diagram of temperature and time, it is natural to consider the temperature as being the result of the time rather than the time as depending upon the temperature. Before starting to draw a graphic diagram the largest and smallest values of the variables should be observed, so that the axes can be located in a satisfactory position on the paper. The next step should invariably be to *lay off a scale along each axis before plotting a single point,* and to see that its zero comes at the " origin " and that *equal numerical intervals are always represented by equal distances* on any one scale, the (positive) x-values always increasing from left to right, and the y-values from below upward. This is especially important when plotting a curve from a table like No. 9 on page 114, where the x-values are given at larger intervals in one part of the table than in another part.

99. Representation of Tabular Values.—After the points corresponding to a set of tabular values have been located on a graphic diagram it is customary to draw a straight line from each point to the adjacent point on the left or right, making a broken line for the " curve " that shows the fluctuations in the value of the dependent variable. If all the points lie on a smooth curve it is

advisable to draw such a curve as evenly as possible, but the experimental values given in a table are apt to show little irregularities which make it impossible to draw a smoothly flowing curve through their graphic points. In such cases the broken line serves the purpose of visually assembling the points in a series, but is not supposed to indicate that intermediate points, if obtainable, would lie exactly along the straight lines.

hour	*temperature*	
	A.M.	P.M.
12	36.9° C.	37.4
1	36.8	37.4
2	36.8	37.6
3	36.6	37.5
4	36.4	37.5
5	36.5	37.6
6	36.6	37.6
7	36.8	37.7
8	36.9	37.6
9	37.1	37.4
10	37.2	37.4
11	37.2	37.2
12	37.4	36.9

NORMAL TEMPERATURE OF THE HUMAN BODY.—Use a horizontal distance of one square to represent one hour, and a vertical distance of one square to represent one tenth of a degree.

Draw a graphic diagram in which the horizontal scale represents the 24 hours of a single day, beginning at midnight and running through twelve o'clock, noon, to the next midnight. For the vertical ordinates use the temperatures given in the table. Connect each point with the next by means of a straight line, being careful not to omit any point. Notice how much more striking and "graphic" the diagram is than the table; how it shows at a glance facts that could be gleaned from the table only with much greater effort.

100. Smoothing of a Graphic Curve.—In the temperature diagram just made it is probable that the little irregularities of temperature are due to accidental causes and would not be exactly repeated in taking the temperature of another individual or even of the same individual on another day. In such cases a more typical picture is given by drawing a smooth, flowing curve in such a way

that it passes through the midst of the scattering points and follows their general upward and downward trend without necessarily cutting most of them or even any one of them.

It ought not to pass below most of the points, nor above most of them, but should leave them distributed, some above and some below, as evenly as possible, subject to the general condition that it must be a smooth curve that does not show even a tendency to resemble a broken line either by indicating the irregularities of the table in the form of wavelets or by suggesting an unduly sharp turn or "corner" at any point.

In another place on the squared paper of the notebook plot merely the *points* as was done for § 99; then draw a *smooth* curve along their general course, outlining it tentatively with a light pencil mark, erasing and correcting this until it is satisfactory, and then tracing it plainly with ink.* It should show not more than one downward loop and one larger upward swing.

This process, which is called *smoothing* a graphic diagram, is advisable only when one can be reasonably certain that the small fluctuations that are eliminated by the process represent unavoidable experimental errors or chance variations or else that they are due to causes which are negligible in the case that is under consideration. Instances have occurred in which even able scientists have missed the discovery of important facts on account of the "smoothing out" of what have seemed to be only accidental irregularities.

101. Questions and Exercises.—1. What shape is the

* Unless special drawing apparatus is used, a curved ink line can usually be drawn more neatly if it is made dotted, as in Fig. 54, instead of solid. The points which it summarizes may be marked rather heavily in order to aid the eye, but this must be done uniformly.

curve of a graphic diagram if each of its points has a y-value that is one half as large as the corresponding x-value?

2. What is the most probable value of the average temperature of a healthy individual at 4 P.M.? Would you prefer to decide the matter from the graphic curve of § 99 or from that of § 100?

3. Can you see any uniformity or law in the arrangement of the points obtained from any of the Tables A, B, C, and D, of § 96? Connect them with smooth, free-hand curves, if this has not already been done. The curve of Table A is called a parabola, B is a straight line, C is a hyperbola, and D is a parabola.

4. Select any two points on the graphic diagram of Table B, and find the difference in their x-values; also the difference in their y-values. Select any two other points on the same line, and treat them in the same way. What relationship always exists between four such differences?

5. How great is the *gradient* (§ 17) of the line represented by Table B? Write an expression for the gradient of any straight line drawn on a graphic diagram in terms of the difference in x-values and the difference in y-values of any two points that are located on the line.

6, 7, 8, 9, 10. Plot the points given in Tables 6, 7 (the numbers that are enclosed in parentheses may be omitted), 8, 9, and 10, using the same scale for the x-axis as for the y-axis, and making each curve large enough to cover nearly the whole page. In each case a perfectly smooth curve can be drawn which will pass through every point. Number 6 is called a *sinusoid* or *sine curve;* no. 7 is a *curve of tangents;* no. 8 is a *parabola;* no. 9 is a *logarithmic curve;* no. 10 is a sinusoid. Extend the curve of no. 10 in both directions as far as the limits of the

paper will allow, using the dimensions of each small ruled square to represent a distance of $\pi/6$ along each axis.

x	y
0.0	.00
0.2	.20
0.4	.39
0.6	.56
0.8	.72
1.0	.84
1.2	.93
1.4	.99
1.6	1.00
1.8	.97
2.0	.91
2.2	.81
2.4	.67
2.6	.52
2.8	.34
3.0	.14
3.2	−.06
3.4	−.26
3.6	−.44
3.8	−.61
4.0	−.76
4.2	−.87
4.4	−.95
4.6	−.99
4.8	−1.00
5.0	−.96
5.2	−.88
5.4	−.77
5.6	−.63
5.8	−.46
6.0	−.28
6.2	−.09
6.4	+.12
6.6	.31

TABLE 6

x	y
0.0	.00
0.2	.20
0.4	.42
0.6	.68
0.8	1.03
1.0	1.56
1.2	(2.6)
1.4	(5.8)
1.6	(−34.)
1.8	(−4.)
2.0	−2.19
2.2	−1.37
2.4	− .92
2.6	− .60
2.8	− .36
3.0	− .14
3.2	+ .06
3.4	.26
3.6	.49
3.8	.77
4.0	1.16
4.2	1.77
4.4	(3.1)
4.6	(8.9)
4.8	(11.)
5.0	(3.4)
5.2	−1.89
5.4	−1.22
5.6	− .81
5.8	− .52
6.0	− .29
6.2	− .09
6.4	+ .12
6.6	.33

TABLE 7

x	y
0.0	4.00
0.4	1.87
0.8	1.22
1.2	0.82
1.6	0.54
2.0	0.34
2.4	0.20
2.8	0.08
3.2	0.02
3.6	0.01
4.0	0.00
4.4	0.01
4.8	0.04
5.2	0.08
5.6	0.13
6.0	0.20
6.4	0.28
6.8	0.37
7.2	0.47
7.6	0.57
8.0	0.69
1.87	0.4
1.22	0.8
0.82	1.2
0.54	1.6
0.34	2.0
0.20	2.4
0.08	2.8
0.02	3.2
0.01	3.6
0.00	4.0
0.01	4.4
etc.	*etc.*

TABLE 8

The curves of Tables 6 and 7 should both be drawn on a single diagram, and the points $\pi/2$, π, $3\pi/2$, and 2π should be marked on the x-axis in addition to the usual scale. The curve for Table 9 also may be drawn on the same diagram; use the y-values as they stand.

x	y
0.1	$-3.+.70$
0.2	$-2.+.39$
0.3	$-2.+.80$
0.4	$-1.+.08$
0.5	$-1.+.31$
0.6	$-1.+.49$
0.7	$-1.+.64$
0.8	$-1.+.78$
0.9	$-1.+.89$
1.0	$0.+.00$
1.2	$0.+.18$
1.4	.34
1.6	.47
1.8	.59
2.0	.69
2.2	.79
2.4	.88
2.6	.96
2.8	1.03
3.0	1.10
3.2	1.16
3.4	1.22
3.6	1.28
3.8	1.34
4.0	1.39

TABLE 9

x	y
0	0
$\pi/6$	$.95\pi/6$
$\pi/3$	$1.65\pi/6$
$\pi/2$	$1.91\pi/6$
$2\pi/3$	$1.65\pi/6$
$5\pi/6$	$.95\pi/6$
π	0
$7\pi/6$	$-.95\pi/6$
$8\pi/6$	$-1.65\pi/6$
$9\pi/6$	$-1.91\pi/6$
$10\pi/6$	$-1.65\pi/6$
$11\pi/6$	$-.95\pi/6$
2π	0
$13\pi/6$	$+.95\pi/6$
$14\pi/6$	$1.65\pi/6$
$15\pi/6$	$1.91\pi/6$
$16\pi/6$	$1.65\pi/6$
$17\pi/6$	$.95\pi/6$
3π	0
$19\pi/6$	$-.95\pi/6$
etc.	*etc.*
$-\pi/6$	$-.95\pi/6$
$-\pi/3$	$-1.65\pi/6$
$-\pi/2$	$-1.91\pi/6$
etc.	*etc.*

TABLE 10

IX. CURVES AND EQUATIONS

Apparatus.—A pencil with a sharp point; pencil compass.

102. Graphic Representation of a Natural Law.—The graphic diagram which is obtained from a table of measurements or experimental data usually shows irregularities, which are sometimes retained by the use of a broken line and are sometimes eliminated by the process known as "smoothing" (§ 100). Neither of these pro-

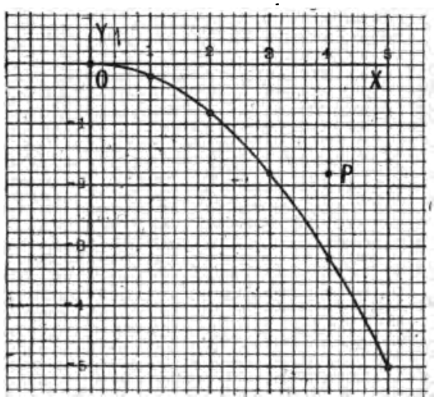


FIG. 24. PARABOLIC PATH OF A FALLING BODY.—The total vertical movement is proportional to the square of the horizontal movement.

cedures is necessary if the variables follow some definite natural law in regard to their changes or if their relationship can be expressed by an equation. In such cases the plotted points in general will lie precisely along a smooth curve without showing any irregularities whatever. For example, a ball that is thrown horizontally will travel,

under the influence of gravity, along a curved path in such a way that its progress in a horizontal direction during 1, 2, 3, 4, $\cdots$ n seconds will be proportional to the numbers 1, 2, 3, 4, $\cdots$ n, while its downward movement will be proportional to the numbers 1, 4, 9, 16, $\cdots$ n^2. If points are so located on a graphic diagram that the vertical distance of each below the x-axis is proportional to the square of its distance to the right of the y-axis (Fig. 24) it will be found that a smooth curve can be drawn so as to pass exactly through all of them, and the relationship between the x-value and the y-value of each and every point will of course be given by the equation $y = -kx^2$ (§ 10). Such a curve is shown more or less steadily by the surface of water that forms a waterfall or by a jet of water that issues from a hose pipe, and is known as a *parabola*.

It will be seen that the scales have been so chosen in Fig. 24 that $k = 1/5$, in other words the equation $y = -kx^2$ has become $y = -\frac{1}{5}x^2$. Test the diagram by assuming that x has the value of 3/2, finding what the corresponding value of y must be by solving the equation, and then locating the point (3/2, $-$ 9/20) which has these two numbers for its x-value and y-value. Does this point lie on the same smooth curve as the points that are shown? Do the same way with each of the x-values 1/2, 5/2, and 7/2.

103. Graph of an Equation.—A single equation that contains two unknown quantities has an infinite number of solutions, for any value whatever may be assigned to one unknown quantity and the corresponding value can always be calculated for the other. Any such solution of an equation that involves x and y will consist of an x-value and a y-value, and so can be represented by the position of a point. All of the infinite number of solu-

tions of a given equation of this sort can theoretically be represented by an infinite number of points on a graphic diagram; thus every point that lies on the curve of Fig. 24 corresponds to two numbers, x-value and y-value, which are a solution of the equation $y = -\frac{1}{5}x^2$. In general, all of the points that represent the solutions of a given equation will be found to lie along a smooth curved or straight line, which is accordingly called the *locus* or "*curve*" *of the equation.* If the locus of an equation extends to an infinite distance, as is the case with the curve of $y = -\frac{1}{5}x^2$, its distant portions are usually more or less flat or straight and uninteresting, so that there is no objection to omitting them from the graphic diagram.

x	y
0	0
1	0
2	6
3	24
4	60
5	120
−1	0
−2	− 6
−3	−24

TABLE OF SOLUTIONS OF THE EQUATION $y = x^3 - x$. The y-values increase so rapidly beyond $x = 5$ that the curve must be nearly straight.

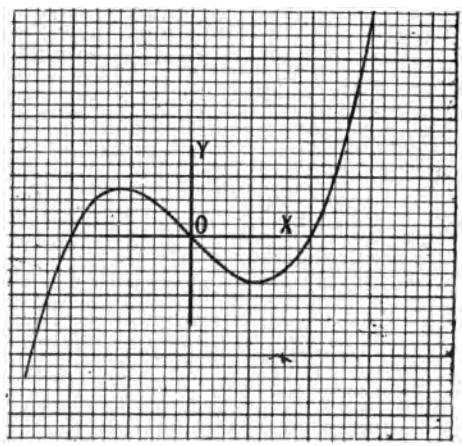


FIG. 25. GRAPH OF THE EQUATION $y = x^3 - x$. Notice that the points $(-1, 0)$, $(0, 0)$, and $(+1, 0)$ are not close enough together to determine the shape of the curve without plotting additional points.

It should be carefully kept in mind that the relationship between an equation and its "curve" or locus means

that *every solution of the equation gives a point which is located on the curve,* and that *every point on the curve furnishes a solution of the equation.* From these two statements it is evident that the coordinates of a point which is not on the curve will not satisfy the equation (try the point (4, $-$ 9/5) shown by the letter P in Fig. 24), and that an incorrect solution (try $x = 2, y = +4/5$) will give a point which does not lie on the curve.

It is obviously impossible to calculate the position of an infinite number of points, but since an equation is known to give a smooth curve it is only necessary to plot enough points to prevent uncertainty as to the shape and position of any part of it.

Make a graphic diagram for the equation $y = x^3 - x$ by the following process: Assume that x has various small integral values, positive and negative, and calculate the corresponding values of y, arranging them in tabular form, as shown on page 117. The x-values have not been carried beyond $+ 5$ on account of the y-values increasing so rapidly. Negative x-values are seen to have the same numerical y-values as positive ones but with the sign changed. The next step is to choose suitable axes and scales (§ 98). Plot the points given by the table before proceeding further. It is not probable that the line has a straight portion between (1, 0) and ($-$ 1, 0) while the rest of it is curved, so it is necessary to calculate at least two more points on the curve, *e. g.*, $x = 1/2$ and $x = -1/2$; and these will be found sufficient to determine a smoothly flowing curved outline.

104. General Procedure.—In plotting the curve of an equation the general procedure is to calculate the table of values, substituting successive positive and negative integral values of x (and fractional values if necessary)

and solving for y; then consider the available space on the paper and draw the axes in a suitable location. Before locating the points that correspond to the tabular values a scale of numbers should always be marked along the x-axis and another along the y-axis. The two scales need not be the same (§ 97), but along each axis, considered by itself, equal distances must always correspond to equal numerical differences, and x-values must always increase to the right and y-values increase upward. If a condensed scale is used it is always advisable to let the dimensions of a small square of the ruled paper represent a simple round number; if the first tabular value is 9200 do not number the successive squares 9200, 18400, 27600, etc., but use the simpler round numbers 10000, 20000, 30000. Where the relative value of the x-unit as compared with the y-unit is unimportant the scales can often be advantageously chosen so as to give the chief part of the curve a slope of about 45°. In the rest of the exercises of this chapter, however, a single square of the paper is to represent a single unit, in each direction, except where otherwise specified.

105. The Straight Line.—The equation $y = 2x + 3$ represents a sloping straight line. For such a simple equation it is hardly necessary to construct a table of values. Locate the x-axis and y-axis in your notebook where there is room for the y-values to extend approximately from $+ 10$ to $- 10$ and for the x-values to extend at least to ± 5. Plot four or five points from solutions of $y = 2x + 3$ obtained mentally, and use a ruler to draw a straight line passing through them and extending a little distance beyond them in each direction. Label this line by writing the equation $y = 2x + 3$ alongside it, close to one end. Without making a new diagram use the same axes and scales for plotting the

following straight lines: $y = 3 - 2x$; $y = 3 + \frac{1}{2}x$;* $y = 3 + 0x$; $y = 2x$; $y = -1 + x$; $y = -1 + \frac{1}{2}x$; $y = -1 - 2x$; $y = -1 - x$. Label each of these in the same way. Examine the lines whose equations have 3 for the numerical term. Do they all pass through the point (0, 3)? Do the lines of the equations that have -1 for a numerical term all pass through the point $(0, -1)$? Which line passes through the origin (0, 0)? Does it seem probable that *the equation* $y = a + bx$ *gives a line that passes through the point* $(0, a)$? Test the point $(0, a)$ by ordinary algebra in order to determine whether it lies on the line $y = a + bx$.

Determine the slope (§ 38 or 17) of the line which you have drawn to represent the equation $y = 2x + 3$, and notice that if the x-value of one point on the line is one unit greater than that of another point on the line then the y-value is always two units greater; furthermore, for *any* two points on the line the difference in y-values is twice as great as the difference in x-values. If the difference in x-values is denoted by Δx and the corresponding difference in y-values by Δy the last statement can be written in the form of an equation $\Delta y = 2\Delta x$ or $\Delta y/\Delta x = 2$.†

In the line representing the equation $y = 3 + \frac{1}{2}x$ does a unit change in x correspond to a change of $\frac{1}{2}$ in y? In $y = 3 + 0x$ does a unit change in x cause no change in y? In $y = 3 - 2x$ does a unit change in x cause an increase of -2 units in y? Can the slope of this last line be considered as equal to -2? Prove algebra-

* Notice that this equation, as it stands, is an explicit statement of the value of y; "clearing of fractions" before starting to substitute would only complicate the work.

† In this notation the symbol Δ is not used to denote an individual quantity or factor but is a "symbol of functionality" like the "log" or "cos" of the expressions $\log x$ and $\cos x$.

ically that when x increases by one unit (*e. g.*, when it changes from m to $m + 1$) the value of y in the equation $y = a + bx$ will increase by b units; thus showing that the equation represents a line whose slope is numerically equal to b. From this it is evident that $\Delta y/\Delta x = b$ for the line $y = a + bx$, and that *the coefficient of x gives the slope of a line if its equation is arranged in the form $y = a + bx$.* Turn to your diagram and see whether the lines $y = 3 - 2x$ and $y = -1 - 2x$ are parallel (*i. e.*, have the same slope). The equation $\Delta y/\Delta x = -2$ is true of both of them and of any other line that runs in the same direction. Just as the line $y = 3 - 2x$ represents all of the infinitely numerous points that lie along its locus (§ 103) and has an infinite number of solutions, so the still more general relationship $\Delta y/\Delta x = -2$ represents an infinite number of parallel lines, any one of which may be considered as a solution of it.

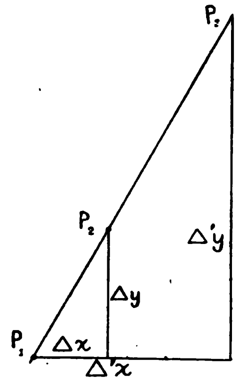


FIG. 26. THE STRAIGHT LINE.—If P_1, P_2, and P_3 are any points on the (curved or straight) line whose equation is $y = a + bx$ it is easy to show algebraically that $\Delta y/\Delta x = \Delta' y/\Delta' x$, and thence geometrically (by similar triangles) to prove that the "curve" is a straight line.

The equation $y = a + bx$ is the general equation for all straight lines except those that run parallel to the y-axis. The latter are given by the equation $x = k$. (Plot the equation $x = 2$ by substituting integral values of y instead of x, and solving for x instead of y.) The most general equation of the straight line is $Ax + By + C = 0$. This can be reduced to the form

$y=a+bx$ if B is different from zero, and to the form $x = k$ if B is zero.

106. The Parabola.—On a single graphic diagram draw and label the curves $y = x^2$, $y = x^2 + 3$, $y = -x^2$, and $y = x^2/10$. The first three of these need not extend beyond $x = \pm 3$, but the last one should be drawn from $x = -10$ to $x = +10$. They are all similar figures and are called *parabolas;* the curves $y = x^2$ and $y = x^2/10$ differ in size but the portion of the latter that extends from -10 to $+10$ is of exactly the same shape as the part of the former that is included between $x = -1$ and $x = +1$. Notice that the curve $y = x^2 + 3$ differs only in position from $y = x^2$; it appears to be the same curve moved three units higher up, and the equations make it clear that for each x-value the y-value of the first curve is greater by three than that of the second.

On another graphic diagram draw and label the three curves $y = x^2 - 3x$, $y = x^2 - 6x$, and $y = x^2 - 7x$. Notice that they are not only of the same shape but of the same size as well.

The general equation of all parabolas whose axis of symmetry is vertical is $y = a + bx + cx^2$. The size of the curve depends only upon the value of c and it is convex downward ("festoon-shaped") if c is positive, convex upward ("arch-shaped") if c is negative. A change in the value of a has been shown to raise (or lower) the curve through a corresponding distance, and obviously it always cuts the y-axis at a height of a. Since any curve cuts the x-axis at the points where $y = 0$ it follows that the parabola $y = a + bx + cx^2$ must cross the x-axis at points whose abscissæ are the two roots of the quadratic equation $0 = a + bx + cx^2$; these roots of course depend upon the values of all three of the parameters a, b, and c.

107. The Probability Curve.—In the chapter on logarithms directions were given for calculating the value of $y = e^{-x^2}$ when x assumes various positive values.* Turn to the table in your notebook that gives the corresponding values of x and y for this equation and plot its graphic diagram on the vacant page next to it. Turn the notebook so that the x-axis will lie lengthwise of the page and use as large a scale as will conveniently permit the x-values to extend from -0.5 to $+2.5$ or 3 (say 20 squares = 1 unit, both horizontally and vertically). For negative values of x notice that the curve is symmetrical with respect to the x-axis; that is, if one point on the curve is $(+0.5, +0.78)$ or (a, b) then another one will be $(-0.5, +0.78)$ or $(-a, +b)$. This could have been inferred from the equation $y = e^{-x^2}$, for x occurs in it only as an even power and any function of $(-a)^2$ must have the same value as the same function of $(+a)^2$.

108. Equation of a Graph.—The process of finding the curve that corresponds to a given equation is not usually difficult, but the reverse operation may be much harder. If the given "curve" is a straight line that cuts the y-axis its equation will be of the form $y = a + bx$, and the values of a and b can be determined according to the two italicized propositions in § 105.

Draw a line through the two points (1, 3) and (2, 1), and find its equation. Ans.: $y = 5 - 2x$. Draw a line through (0, 2) and (3, 0), and find its equation. Find the equation of a line drawn through $(0, -3)$ and

* Although an equation is properly a *sentence* the student will sometimes find it apparently used as a *noun*. In such cases one side of the equation is to be considered as the substantive and all the rest of it as a parenthetical statement; thus the expression in the text means, "for calculating the value of y (*which is the same thing as* e^{-x^2}) when x assumes. . . ."

(5, 0). See if your last two equations can be transformed into $x/3 + y/2 = 1$ and $x/5 + y/(-3) = 1$, respectively. What is the apparent significance of the denominators m and n in the equation $x/m + y/n = 1$?

If the given curve is a parabola it has been seen that its equation will be $y = mx^2$ *if the origin is at the vertex of the curve.* Turn to your plot of $y = x^2 - 6x$ (§ 106) and draw a new pair of axes through the point (3, − 9), which is the vertex of the curve. If the coordinates of points on the curve, referred to these new axes, are denoted by the capital letters X and Y, notice that the same curve corresponds exactly to the relationship $Y = 1X^2$. From this equation it is possible to deduce the original equation as follows: Let P be *any* point on the curve; notice that its X-value is 3 less than its x-value, also that its Y-value is 9 more than its y-value. That is, for every point on the curve the equations $X = x - 3$ and $Y = y + 9$ hold good, as well as $Y = 1X^2$. Substituting in the last equation the values of X and Y given by the other two, $y + 9 = (x - 3)^2$ or $y = x^2 - 6x$.

Find the equation of the parabola that was drawn for Table A, § 96.

With a pencil compass draw carefully a circle whose radius is 5 and whose centre is the point (4, 3) on a graphic diagram. The curve will be seen to pass exactly through the points (0, 0); (1, − 1), (4, − 2), and three corresponding integral points in each quadrant. Mark any point P on the circumference, and considering the centre of the circle as a new origin draw the ordinate Y of the point P and from its base draw the abscissa X along the new X-axis. From the theorem of Pythagoras and the fact that all radii are equal the coordinates of the point P must satisfy the relationship $X^2 + Y^2 = 5^2$; and since $X = x - 4$ and $Y = y - 3$ this relationship be-

comes $(x - 4)^2 + (y - 3)^2 = 5^2$, or $y = 3 + \sqrt{25 - (x - 4)^2}$, which is the equation of the given circle.

109. Change of Scales.—Lay off an x-scale in which three squares of the ruled paper correspond to each unit and a y-scale of two squares per unit, and plot the circle $x^2 + y^2 = 25$ from the following twelve points $(0, \pm 5)$, $(\pm 3, \pm 4)$, $(\pm 4, \pm 3)$, $(\pm 5, 0)$, drawing as smooth a curve as possible. The result is an *ellipse*, or a "strained" (*i. e.*, distorted) circle.

To find the equation of such an ellipse, *referred to uniform scales*, let each square of the ruled paper be considered as equal to unity. Then each point on the ellipse will be twice as high or low and three times as far to the right or left as a corresponding point on the true circle $x^2 + y^2 = 25$. That is, if the coordinates of a point on the ellipse are called X and Y, then $X = 3x$ and $Y = 2y$; in other words, $\frac{1}{3}$ of the X-value and $\frac{1}{2}$ of the Y-value will satisfy the relationship for the circle, so that $(X/3)^2 + (Y/2)^2 = 25$. This is an illustration of the general fact that re-writing an equation with x/a and y/b in place of the original x and y will stretch out the length and height of the diagram to a and b times the original dimensions, respectively.

On one graphic diagram plot the loci of $x + y = 1$ and $x/3 + y/5 = 1$.

110. Definitions of Circular Functions. —The use of a graphic diagram makes it possible to give very concise definitions of the circular functions of an angle of any size: Let a line that coincides with OX (Fig. 27) rotate counterclockwise through an angle a to a new position OP. If the coordinates of the point P are denoted by x and y, and its distance from the origin by $+ r$, then $\sin a = y/r$, $\cos a = x/r$, $\tan a = y/x$, $\cot a = x/y$, $\sec a = r/x$, and $\csc a = r/y$. These relationships hold

true whether the angle is larger or smaller than 360°, and for negative angles when described by a clockwise rotation. They furnish the easiest method of determining the *sign* of a circular function; in Fig. 28 ($< 103°$) the

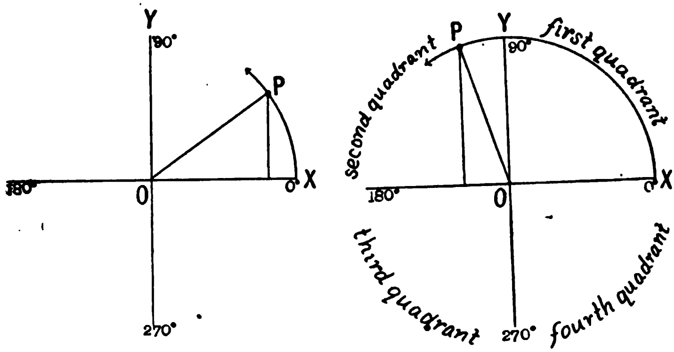


FIG. 27. FUNCTIONS OF AN ANGLE.—For an angle of any magnitude, described counterclockwise from OX, the sine, cosine, tangent, cotangent, secant, and cosecant are respectively y/r, x/r, y/x, x/y, r/x, and r/y, where r is the distance from the point (x, y) on the rotating line to the origin.

value of y is positive but x is negative; r is always considered positive, so it is immediately obvious that the sine of an angle "in the second quadrant" (between 90° and 180°) is positive while the cosine is negative.

111. Questions and Exercises.—1. If one point on the curve of $y = x^3 - x$ (Fig. 25) is (a, b) prove that another one is $(-a, -b)$.

2. Knowing the shape of the graph of $y=a+bx+cx^2$, what facts can you deduce in respect to the graph of $x = a + by + cy^2$?

3. What effect will be produced if the equation of a curve is re-written with $x + p$ and $y + q$ substituted everywhere for the original x and y? (Try the parabola,

$y = x^2$, and $p = 0$, $q = 3$, if there is any uncertainty.)

4. What effect will be produced if the equation of a curve is re-written with mx substituted everywhere for the original x? If my for the original y?

5. Make a graphic diagram of the equation $y = 30/x$. Plot enough points to show clearly the form of both parts of the curve. On the same diagram plot $y = 1/x$. Each of these curves is a (*rectangular*) *hyperbola.*

6. Plot the curve of $y = 1/x^2$ carefully. How would the appearance of the curve $y = 30/x^2$ differ?

7. Solve graphically the two simultaneous equations $x^2 + y^2 = 25$ and $y = x^2 - 3x$ by drawing both curves on one diagram and locating their points of intersection.

8. Plot the curves of one or more of the following equations: (a) the "*semi-cubical*" parabola $y^2 = x^3$; (b) the finite curve $y^2 = x(10 - x)^3$, using a condensed scale along the y-axis; (c) the curve $y = x \log x$; (d) $x^2 - y^2 = 0$; (e) $x^2 + y^2 = 0$.

9. Instead of being located by latitude and longitude (*rectilinear coordinates*, referred to an x-axis and a y-axis), the position of a point may be located by its distance and direction, *i. e.*, the length r and the angle a of § 110 (its *polar coordinates*, referred to a *pole* and an *initial line*). Obviously, $\tan a = y/x$ and $r^2 = x^2 + y^2$; or $x = r \cos a$ and $y = r \sin a$. The distance and angle are called *polar coordinates* and are usually indicated by the letters r (radius) and θ (angle), or by ρ and θ.

Make a rough graphic diagram of the curve of the equation $x^2 + (y - h)^2 = h^2$ (a circle that passes through the points $(0, 0)$, $(0, 2h)$, and $(\pm h, + h)$; compare § 108); then substitute $x = \rho \cos \theta$, $y = \rho \sin \theta$ and show that the *polar equation* of this circle is $\rho = 2h \sin \theta$.

What must be the general shape of the curve $\rho = 2\theta$? Of (a) $\rho = 2$; (b) $\rho = 1/\cos \theta$ or $\rho = \sec \theta$; (c) $\theta = 1$.

X. GRAPHIC ANALYSIS

Apparatus.—Fine black silk thread; slide rule; graduated ruler; pencil with a sharp point.

112. Interpretation of Equations.—Turn back to the diagrams that were drawn to represent the equations $y = 2x$ and $y = 2x + 3$ (§ 105) and notice that for every point of the first there is another point just three units higher on the second of the two straight lines. The equation $y = 2x + 3$ makes the statement that the value of y is as much as the $2x$ of the first equation with three more added. Another way of looking at the same equation can be shown by writing the terms in reversed order, $y = 3 + 2x$. This can be considered as making the statement that, for every value of x, y is equal to the amount 3, to which there is further added the amount $2x$ (Fig. 28). It is worth while to form the habit of always investigating the meaning of an equation as far as possible, especially for the student who intends to proceed further with the study of any of the practical applications of mathematics. For example the equation, $s = v_0t + \frac{1}{2}at^2$, of uniformly accelerated motion

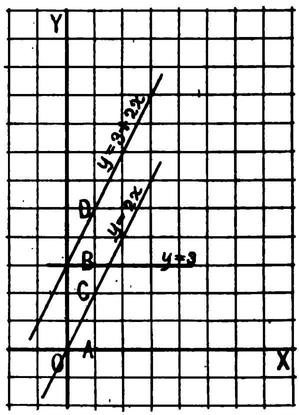


FIG. 28. GRAPH OF THE EQUATION $y = 3 + 2x$.—The ordinate of any point (D) on the line may be considered as being either 3 more than $2x$ (CD added on to AC) or $2x$ more than 3 (BD added on to AB).

means that the space traversed is equal to the space, v_0t, that would have been traversed by an object moving at an initial velocity of v_0 without acceleration, plus the $\frac{1}{2}at^2$ of distance that a stationary body would have been made to describe if acted upon by the accelerating force alone—an illustration of the addition of vectors, which in this case are directed along the same straight line. The student of physics will find no difficulty in further analyzing each of these two separate terms.

113. The Graph of $y = a + bx$.—It has already been seen (§ 105) that the equation $y = a + bx$ has a straight line for its "curve," and that this line cuts the y-axis at a height of a and has a slope that is numerically equal to b. For this reason it is often spoken of as a *linear equation*, and the law of variation which it illustrates is known as the "*straight line*" *law*. It should be noticed that the values of x and y are not proportional except for those cases in which the straight line passes through the origin.

Draw the loci of the following equations *without calculating* the values of x and y for any point. For each one rule a straight line in such a position that it intersects the y-axis at the required point and has the required gradient. If there is any doubt as to the meaning of a negative value for the b of the equation $y = a + bx$ a few points may be calculated from the equation (5) in the usual manner. (1) $y = 2 + 2x$. (2) $y = 2 + x$. (3) $y = 2 + \frac{1}{2}x$. (4) $y = 2 + 0x$. (5) $y = 2 - \frac{1}{2}x$. (6) $y = 4 - 2x$. (7) $y = 4 - x$. (8) $y = 0 - \frac{1}{2}x$. (9) $y = \frac{1}{2}x$. (10) $y = -2 - x$. (11) $y = -2 + x$. (12) $y = -2$.

Lay the ruler on the squared paper at random in any position and draw a straight line. Mark the axes along any convenient ruled lines of the paper and determine the

equation of the line that was drawn, expressing the numerical term and the coefficient of x as decimal fractions.

114. The Straight Line Law.—If a homogeneous metal bar is heated it may be expected to expand in such a way that its length is always proportional to its thickness, and if these two variables are denoted by x and y the relation between them will be given by the equation $y = kx$, in which k has a constant value. If we compare length and temperature, however, there is no proportionality between them. The bar is not made twice as long by heating it twice as hot, but it is not difficult to show experimentally that there is certain law of relationship, namely, the *change* in length is proportional to the *change* in temperature. If a bar whose length is 10 at a temperature of 15° C. is expanded to 12 at 20° C., then its length at 30° C. will be 16.* In other words, $12-10 : 16-10 :: 20-15 : 30-15$. Draw a graphic diagram to illustrate this example, plotting temperature horizontally (§ 98) and length vertically, and notice that the points (15, 10), (20, 12), and (30, 16) lie in the same straight line. The variables x and y are not proportional; it is only their respective changes or differences that show proportionality. Using the notation of § 105, ¶ 2, we can write $\Delta y : \Delta x :: \Delta' y : \Delta' x$, or $\Delta y/\Delta x = k = 2/5$ for the graph (draw these Δ's on your diagram), and for the general physical law $\Delta(length)/\Delta(temp.) = k$. When two variables show proportional changes they are said to follow a *linear law*, or a *straight line law*.

* These numbers are very greatly exaggerated as compared with those for ordinary materials. Metals expand only a few hundred-thousandths of any dimension when heated a few degrees. Furthermore the law is only approximate; careful experiments on a given bar will show that its length at t° C. will not be exactly expressible in the form $l = l_0(1 + kt)$, but will need a more complicated equation, $l = l_0(1 + k_1t + k_2t^2)$ or even $l = l_0(1 + k_1t + k_2t^2 + k_3t^3)$.

Prove (*a*) that proportionality is always in accordance with the straight line law, but (*b*) that a straight line law does not mean proportionality between the two variables except when the straight line *passes through the origin.*

115. The "Black Thread" Method.—If a set of experimental measurements, such as those of temperature and length of a metal bar, are found to correspond approximately to a straight-line law they may be plotted as the x's and y's of a graphic diagram, and their irregularities may be eliminated ("smoothed") by drawing the straight line that appears to come closest to all of the points. This is called the *black thread method* because the position of the line is decided by making use of a stretched thread instead of a ruler; the thread and the points can all be seen at the same time, while a ruler would hide half of the points if properly placed.

Plot the values given in the table as accurately as possible, marking each point by a minute dot surrounded by a small circle, or by a cross composed of a short vertical mark to indicate the exact value of the abscissa and a short horizontal line at the exact height of the ordinate.

x	y
1	9.8
2	8.5
3	8.0
4	7.2
5	6.7
6	6.5
7	6.2
8	5.5
9	5.0
10	4.1
11	3.9
12	3.2
13	2.3

EXPERIMENTAL DETERMINATIONS OF LINEAR VARIATION.

Be sure that the page of the notebook rests in a perfectly flat position and stretch a fine black silk thread on it in such a position that it follows the general direction of the points. Move it a trifle toward the top or bottom of the page, also rotate it slightly, both clockwise and counter-clockwise. Attempt to get it into such a position that it lies among the points like a smoothed curve (§ 100), following their general trend but not

necessarily passing exactly through any one of them. See that there are about as many points above the line as below it, but if the high points are more numerous toward one end of the thread and the low ones toward the other end rotate the thread enough to remedy the condition. When the thread is finally arranged in the most satisfactory position do not attempt to draw the line but *notice* where the thread cuts the x-axis and where it cuts the y-axis. From these two numbers calculate the slope (gradient) of the thread, noticing whether its value is positive or negative. Write the equation of the line that is indicated by the thread, making y equal to a numerical value plus a certain number of times x; *i. e.*, write the equation in the form $y = a + bx$ (§ 105).

116. Intercept Form of a Linear Equation.—The equation $x/m + y/n = 1$ must be the equation of a straight line, since it is easily reducible to the form $y = a + bx$. Substitute zero for the value of x and notice that the corresponding value of y is n. Show likewise that when y is equal to zero x will be equal to m. In other words the graph of $x/m + y/n = 1$ passes through the points $(0, n)$ and $(m, 0)$ (compare § 108, ¶ 2).

Draw the straight line $x/(-3) + y/2 = 1$ by ruling a line through $(-3, 0)$ and $(0, 2)$; then reduce the equation to the form $y = a + bx$ and see whether the coefficients a and b verify the y-intercept* and the gradient of the ruled line.

Since *m and n are the x-intercept and y-intercept of the line $x/m + y/n = 1$* the equation of a line that cuts both axes can be written immediately without any calculation. Write the equation of your black thread determination in this form.

* The points at which a locus cuts the x-axis and the y-axis are called its *x-intercept* and *y-intercept*, respectively.

117. The Graph of $y = a + bx + cx^2$.—Just as the curve $y = a + bx$ can be considered as having its ordinate for each value of x built up of the ordinate a plus the ordinate bx (§ 112, Fig. 28), so the more elaborate equation $y = a + bx + cx^2$ can be considered as representing a curve which is made by piling up the parabola $y = cx^2$ upon the slanting line $y = a + bx$. Curiously enough this also represents a parabola in every case in which c is different from zero; the slant of the straight line does not cause the curve to be lop-sided.

On a single graphic diagram plot both $y = -0.1x^2$ and $y = 3 + 0.5x$. To the ordinates of the latter add (or subtract, as the case may require) the ordinates of the former for each integral value of x, and draw as smoothly as possible the resultant curve of the equation $y = (3 + 0.5x) + (0.1x^2)$.

If a curve that is obtained from experimental measurements looks like a portion of a parabola it is possible to draw an approximate tangent, find its linear equation, and then determine a value of c that will make the equation $y = a + bx + cx^2$ fit the given curve. An algebraical procedure that accomplishes the same result is to measure the coordinates of some point on the curve (say (2, 3)) and substitute in the general equation (giving $3 = a + b \times 2 + c \times 2^2$); repeating this with two more points gives three equations, from which the values of the unknown $a, b,$ and c may be determined. For a graphical procedure it is usually more satisfactory to complete a free-hand parabola as far as its vertex, if that is not already present, and then continue as in the following example.

118. Law of Density-Variation for Water.—The density of water at different temperatures is given in the table. If the density is called y (§ 98) and the temper-

ature x, it is required to find the numerical values of the coefficients of the equation $y = a + bx + cx^2$ that will express the law of variation.

The first step is to make a careful graph. Use extended scales for both x-values and y-values so that the diagram will cover practically a whole page of your notebook, noticing that the y-values need not include zero but only extend from .995 near the bottom of the page through .996, .997, .998, and .999 to 1.000 near the top. The curve will be seen to have the appearance of an "arch-shaped" parabola (§ 106), so that it is evident that its equation will be approximately $y = -mx^2$ if the origin is located at the vertex of the curve, namely at the point (4, .99997). With the origin in this position the ordinate for 4° will obviously be zero, for a temperature 2° higher than this the ordinate will be $-.00003$ (*i. e.*, .99997 − .99994), for 4° higher it will be − .00012, etc. These y-values and x-values have been given in the second table, where for simplicity the negative signs and the decimal points have been omitted.

temp.	*dens.*
0	.99984
2	.99994
4	.99997
6	.99994
8	.99985
10	.99970
12	.99950
14	.99924
16	.99894
18	.99859
20	.99820
22	.99777
24	.99730
26	.99678
28	.99623
30	.99564

x	y	$\sqrt{y}$
0	0	
2	3	
4	12	
6	27	
8	47	
10	73	
12	103	
14	138	
16	177	
18	220	
20	267	
22	319	
24	374	
26	433	

RELATIONSHIP BETWEEN TEMPERATURE AND DENSITY OF WATER.

If the values in the second table correspond to an equation of the form $y = kx^2$ the square root of y must be proportional to x itself. Using the slide rule, read off the square roots of the numbers in column y and enter

them in the vacant column headed $\sqrt{y}$. Since the x's in the first column and the $\sqrt{y}$'s in the third one are proportional their relative magnitudes can be determined by the black thread method.

Plot their values on another graphic diagram, unless special directions to the contrary are given by your instructor, and determine the slope with the black thread. Since this is a case of proportion the thread must of necessity pass through the origin (§ 114, ¶ 2), even though it may appear to lie less evenly along the row of points than it otherwise would. (Measurements which must necessarily fulfill a certain conditional relationship are called *conditioned measurements* and will be considered later. In this case notice that the temperature and density of water are not in themselves conditioned measurements, but we are attempting to make them satisfy a "*condition*," namely, that they shall follow the law $y = a + bx + cx.^2$) The value obtained for the slope of the black thread will probably be in the neighborhood of 5/6 or 0.83.

x	$.69x^2$	$y + 3$
0	0	3
2	3	6
4	11	14
6	25	28
8	44	47
10	69	72
12	99	102
14	135	138
16	177	180
18	224	227
20	276	279
22	334	337
24	398	401
26	467	470

TABLE OF VALUES FOR $y = .69x^2$ AND FOR $.69x^2 + 3$.

If $\sqrt{y} = .83x$, then, it follows that $y = .69x^2$, y being expressed in hundred-thousandths as in column 2 of the second table and being measured downward from the level of the vertex of the parabola. Instead of measuring y-values downward from the original 99997 it will be found easier to increase them by 3, as in the third column of the next table, and then measure the results downward from the level 100000, which is probably one of the ruled lines of the plotting paper.

On the graphic diagram already used for plotting the density of water lay off the values of $y + 3$ (in .00001's of a unit of density proper) downward from the line $y = 1.00000$, noticing particularly that $x = 0$ is now at the vertex of the curve (at 4°, not at 0°). If the work has been carefully done it will be seen that this new curve of the equation $y = -.69x^2$, or more accurately, $100000y^* = -.69x^2$ forms a fairly good approximation to the unknown relationship of the empirical values of density and temperature.

The last step that remains to be taken is to reduce the equation $100000y = -.69x^2$ to the original axes of temperature and density. Since x is zero when t (temperature) is 4, and 2 when t is 6, etc., it is plain that $x = t - 4$ everywhere; in the same way $y = d - 0.99997$. Substituting these values in the original equation $100000y = -.69x^2$ gives

$$d = 0.99986 + 0.0000522t - 0.0000069t^2$$

which is the required law expressing the relationship between density and temperature.

119. Typical Curves.—Any law of change (as far as finite values of the variables are concerned) can be expressed by an equation of the form $y = a + bx + cx^2 + dx^3 + ex^4 + \cdots$ if enough terms are used, but it will easily be understood that the method soon becomes difficult to handle. Sometimes the appearance of the curve makes it possible to guess that its equation is of some particular form (Figs. 29–34), or the form may be deducible from theoretical considerations (*cf.*

* The large coefficient is needed because the y-values do not properly run into the hundreds of units as would appear from the table but are condensed within a small fractional range (.99564 to .99997) if they are to represent densities correctly. See § 109, and § 111, No. 4.

Fig. 33). For a curve that deviates only slightly from a straight line it often happens that the black thread method will give a linear law that is a sufficiently good approximation for practical purposes. The parabola can usually be fitted fairly well to a curve that shows a single upward or downward sweep, and is easier to apply than the exponential curve, which is so often used instead.

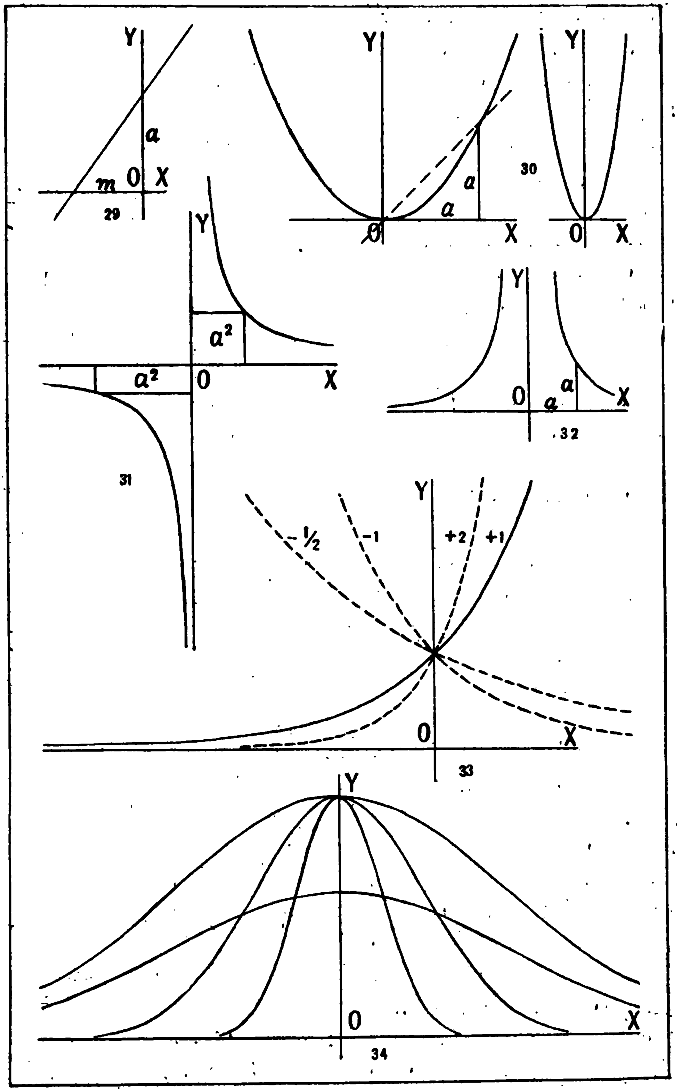
120. Linear Relationship by Change of Variables.—In the previous section the values of y were not proportional to those of x, but a straight line was obtained by plotting x and $\sqrt{y}$ instead of x and y. A transformation of this sort can always be made when the type of equation has been picked out and its numerical constants are to be determined. Thus if $y = e^{ax}$ the transformed equation $\log y = ax \log e$ shows at once that x and $\log y$ are proportional; if $y = ax/(b + x)$ it will be found that y and y/x follow the straight line law; if $y = a/x$ either of the variables will be directly proportional to the reciprocal of the other; etc.

The volume of a certain mass of air was found to vary, under changes of pressure to which it was subjected, according to the numbers in the following table. Assuming that the pressure and volume are inversely proportional, represent the relation between them by an equation after plotting a smoothed curve on a graphic diagram with a condensed x-scale (pressures) and an expanded y-scale (volumes):

pressure	*volume*
760 mm Hg	8.1 cm³
830	7.2
926	6.6
1022	5.8
1125	5.3
1230	4.9
1340	4.5
1410	4.1
1520	4.0
1600	3.9

EXPERIMENTAL DETERMINATION OF pv-VARIATION OF A GAS.—The law of variation for a constant mass of gas is known to be of the form $v = k/p$.

Since y is inversely proportional to x a new variable,


Y
a
m
O
X
29
Y
a
a
O
X
30
Y
O
X
Y
a²
a²
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31
Y
O
a
a
X
32
Y
-½
-1
+2
+1
O
X
33
Y
O
X
34

$1/y$, can be obtained, which will vary directly with x. By plotting x and $1/y$ and holding a black thread so as to pass through the origin (§ 114, ¶ 2) find an equation connecting x and $1/y$, and reduce it to a simple equation between x and y. The result will probably be in the neighborhood of $v = 6000/p$. Plot this equation on the same graphic diagram as the smoothed curve and notice how closely the two loci correspond.

FIG. 29. THE STRAIGHT LINE.—Its equation may be written either $y = a \pm (a/m)x$, taking care that the right sign is used for the gradient; or $x/m + y/a = 1$, if the signs of m and a are taken according to the usual convention.

FIG. 30. THE PARABOLA $ay = x^2$.—The equation shows that it cuts the line $y = x$ at the point (a, a). While running infinitely far upward the curve also extends infinitely far to the right or left, but approaches verticality, and its ends subtend an angular distance that, seen from the vertex, approaches zero. All parabolas are similar figures.

FIG. 31. THE RECTANGULAR HYPERBOLA $xy = a^2$.—The curve consists of two separate parts ("branches"), each of which extends to an infinite distance and approaches two fixed lines, called its *asymptotes*, without ever reaching them. All *rectangular* hyperbolas are similar figures.

FIG. 32. THE CURVE $y = a^3/x^2$.—Its equation shows that it passes through the points $(\pm a, + a)$, $(\pm 0, + \infty)$, and $(\pm \infty, + 0)$, and that it is symmetrical with respect to the y-axis since there is a point $(- m, + n)$ corresponding to every point $(+ m, + n)$.

FIG. 33. THE EXPONENTIAL CURVE $y = e^x$.—A remarkable property of this curve is that its slope is everywhere equal to its ordinate at the corresponding point. In the more general equation $y = e^{mx}$ the slope is *proportional* to the ordinate, so that the curve may be used to represent a relationship like *Newton's Law of Cooling; viz.*, the rate at which a body loses temperature is proportional to the temperature itself. Curves for $m = 2$, $m = - 1$ and $m = - 1/2$ are shown by dotted lines.

FIG. 34. THE CURVE OF ERRORS $y/b = e^{-(x/a)^2}$.—Three different values of a are represented for a large value of b and one of a for a smaller one of b.

121. Questions and Exercises.—1. With the black thread find the best straight-line approximation for $v = 6000/p$ or for your smoothed curve. Ans.: approximately $y = 7.7 - .47x$.

2. Name the kind of curve that would correspond to the equation obtained by solving

$$\begin{cases} 8.1 = a + b(760) + c(760)^2, \\ 5.3 = a + b(1125) + c(1125)^2, \\ 4.0 = a + b(1520) + c(1520)^2, \end{cases}$$

for a, b, and c, and substituting the values so obtained in the equation $y = a + bx + cx^2$. For what purpose could the resultant equation be used?

3. Explain how an equation of the form $y = e^{ax}$ could be determined for the smoothed curve of the last section.

4. Show that the distance between the points (3, 4) and (7, 5) is equal to $\sqrt{(3-7)^2 + (4-5)^2}$, and write a general formula for finding the distance between any two points, such as (x_1, y_1) and (x_2, y_2).

5. The equation of the straight line that passes through the two points (x_1, y_1) and (x_2, y_2) is

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y - y_1}{x - x_1}.$$

Draw a diagram of the line and the two points, and explain the significance of the subtracting, the dividing, and the equating in the above formula.

6. Guess at the equation of table 9 in § 101. Test the equation by substituting a few tabular values, and if your first estimation was faulty, make a better one.

7. Prove that every point on the parabola of Fig. 30 is as far from the line $y = -a/4$ as it is from the point

$(0, + a/4)$. The line is called its *directrix* and the point its *focus*.

8. If e^{ax} is always identical with $(e^a)^x$ and if $2.718^{2.303}$ is equal to 10, turn to Fig. 33 and see how the curve $y = 10^x$ would run. (The reciprocal of 2.303 is .4343.) Decide how the curve $x = 10^y$ would compare with it. Have you drawn the latter curve previously?

XI. INTERPOLATION AND EXTRAPOLATION

Apparatus.—Black thread; slide rule; pencil with a sharp point.

122. Definitions.—The process of drawing the locus of an equation by plotting a few isolated points and filling up the intermediate positions with a smooth curved line involves the tacit assumption that the values of y for the intervening values of x would have been found to vary in this predeterminate manner if they had been calculated. That is, the value of a function that corresponds to a certain magnitude of its independent variable need not be obtained by calculation in all cases, but may often be determined by comparing it with the values which the function is known to have when the variable is larger and smaller than in the particular case that is under investigation. When a few known values are used for the purpose of determining an intermediate unknown value the latter is said to be found by *interpolation*. For example if the population of a city were known for each of the years 1850, 1860, 1870, 1880, 1890, and 1900, one could guess fairly accurately what the population amounted to in the year 1875, even if the given values should not follow any known law or any recognizable type of curve.

The process of using a certain range of values for determining a value that lies outside of that range is called *extrapolation* (Latin, *extra*, outside; *inter*, between; *polire*, to make smooth). Thus, from the data mentioned above it would be possible to make some kind of an estimate of the population for the year 1840, or for 1910, or perhaps even for 1920.

Turn to your diagram of the daily variations in body temperature (§ 99) and determine the normal temperature of the human body at 9:30 A.M. It should be about 37°.07, .08, or .09. What was the temperature of this individual at 9:30 A.M. on the day of the experiment. Ans.: probably about 37°.15 or 37°.16.

Interpolation is a process that is trustworthy only when the data are sufficiently numerous and are given at sufficiently close intervals, and when their variation is not too irregular. It would be impossible to interpolate the values $y = x^3 - x$ (Fig. 25) from the three points $(-1, 0)$, $(0, 0)$, $(+1, 0)$; or to fill in a free-hand parabola for $y = x^2 - 2x - 3$ if the only data were the points $(-1, 0)$ and $(+2, -3)$.

123. The Principle of Proportionate Changes.—It is always necessary to make an assumption of some kind when a process of interpolation is used. In obtaining logarithms from a table by interpolation it is assumed, for example, that the logarithm of 2718 is 8 tenths of the way from log 2710 to log 2720 (§ 66), or in general that any change in a logarithm is proportional to the change in the corresponding natural number. This is an instance of the *linear law* (§ 114), and may be expressed as $\Delta(\log x) = k\Delta x$. It is true only for *small* differences (log 300 − log 200 is not equal to log 200 − log 100), because the curve of $y = \log x$ is nowhere nearly a straight line unless a very small stretch of it is considered by itself.

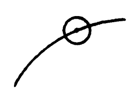


FIG. 35. ELEMENTARY STRAIGHTNESS.—When a circle that is drawn around any point on a *smooth* curve is made smaller and smaller it is cut more and more nearly into exact halves by the curve itself.

Plot a graphic diagram of $y = \log x$ from $x = 0$ to $x = 10$ using a large scale on the x-axis (5 □'s = 1 unit)

and a much larger one on the y-axis (20 □'s = 1 unit). Do *not* use any x-values except 0.1, 0.2, 0.4, 0.6, 1.0, 2.0, 3.0, $\cdots$ 10.0. Draw a free-hand curve as smoothly as possible through the corresponding points. *Measure* the y-values for $x = 3.5$, 2.5, 1.4, and 0.8; then verify each by referring to the tables. Notice that the points on the curve for which $x = 1, 2, 3$, do not lie in a straight line. Notice that the short stretch of curve that includes the points $x = 2710$, 2718, 2720 has no perceptible curvature. If the points are specified with sufficient accuracy, however, it will be found that no three of them lie in the same straight line; thus four-place logarithms may be safely interpolated if the values for every three-figure natural number are given, but five-place accuracy for the logarithms necessitates four-figure values for the natural numbers in the first part of a logarithm table, that corresponds to the more sharply curved part of the graph (compare tables in appendix).

The fact that a curve is "smooth" means that a very short stretch of any part of it deviates very little from a straight line. In other words, if a small circle is described around any point of such a curve its circumference will be cut by the curve at two points whose angular distance approaches 180° as the circle is made smaller and smaller (Fig. 35). This property of a curve is known as *elementary straightness*, the term "element" being used in the sense of "a very small portion," and is characteristic of the curves of all equations in which y can be expressed as a rational function of x. Some "transcendental" equations* lack elementary straightness at a few points, but in general the process of linear inter-

* A *transcendental* equation is one that involves non-algebraical functions; for example, $y = x \log x$. The curve of this equation cuts the elementary circle around (0, 0) at only *one* point, *i. e.*, comes to an abrupt end at the origin.

polation is applicable to any tabular values that are given at sufficiently small intervals.

124. Examples of Linear Interpolation.—Consult the table in § 118 and state the density of water at 21° C.

Water boils at 100° C. when the barometric pressure is equal to that of a 760-mm. column of mercury; to make it boil at 101° C. the pressure must be raised to 788 mm. Hg. What should an accurate thermometer register in boiling water when the barometer stands at 775?

Plot a few points for the equation $y = x^2 - 2$ and fill out a free-hand curve. When x is 1 y is -1; when x is 2 y is $+2$. Accordingly, if the locus were not curved it would intersect the x-axis at $x = 1\frac{1}{3}$. Substituting $x = 1.3$ gives $y = -.31$; substituting $x = 1.4$ gives $-.04$; substituting $x = 1.5$ gives $+.25$. The stretch of the curve that lies between 1.4 and 1.5 (Fig. 36) is so nearly straight that the point where it intersects the x-axis can be found quite accurately by the law of proportionality: $\Delta y = .29$ for $\Delta x = 0.1$, so Δy should equal the .04 required to bring y up to zero for .04/.29 of 0.1, or .014, beyond 1.4. Substituting $x = 1.414$ gives $y = -.000604$; substituting $x = 1.415$ will be found to give $y = +.002225$. Another application of the principle of proportionality gives $x = 1.414 + .000214$ or 1.414214, and thus the calculation of the value of $\sqrt{2}$

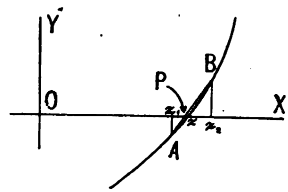


FIG. 36. SOLUTION BY "DOUBLE POSITION."—Any equation can be solved approximately by aid of a graph. Then by using two x-values that are near the required root, x, and interpolating a y-value along a straight line it is possible to find as close an approximation, P, as is desired.

proceeds with continually increasing rapidity. This, of course, is a determination of the value of one root of the equation $0 = x^2 - 2$, and is known as the method of *false position*, or of *double position*. It can be used for any equation whatever containing one unknown quantity by arranging it in the form $\phi(x) = 0$* and drawing a graphic diagram of $y = \phi(x)$.

A delicate beam balance has a pointer which swings to a position of rest at 8.0 on an arbitrary scale when an unknown mass is balanced against standard weights of 14.837 grams but stops at 10.5 when the weights are changed to 14.836. What weights would be required to bring the pointer to the central position, which is at 10.0 on the scale? Ans.: $14.837 - \frac{4}{5}(.001)$; or 14.8362.

A telephone company charges for measured service at the yearly rates given in the table. It will be plain that an increase of 200 messages means an additional cost of nine dollars, making the rate for an increased number of messages equal to \$9/200 m, or 4.5c. per message. At this rate 600 messages should cost 27 dollars; accordingly, it is evident that the rate for any number of messages is made up of two parts, a flat charge of \$21 plus a message rate of \$9 per hundred. If the number of messages is represented by x and the total cost by y, then the equation connecting them will be $y = 21 + .045x$, showing straight-line variation without proportionality.

no. of messages	*charge*
600	\$48
800	57
1000	66
1200	75

EXAMPLE OF A "READINESS-TO-SERVE" CHARGE INCORPORATED WITH A RATE CHARGE.

* The expression $\phi(x)$ means "any function of x," and may be used as a general form to denote $3x^2$, $\log x$, $2a^x$, or any other function, just as the general symbol m may be used to stand for the number 2, or 100, or any number whatever. As alternative notations $f(\ )$, and $F(\ )$ are often used.

125. Graphic Interpolation.—In general, tabular values do not follow the straight-line type of variation, and rather complicated formulæ may be required for purposes of interpolation. In case a graph can be drawn, however, there is usually no difficulty in constructing a smooth curve (or "smoothed" curve, as occasion may require) and obtaining any intermediate values simply by measuring them on the graph. The process is sometimes uncertain or erroneous if the given values are not close enough together or if their variation is too irregular. It will have been noticed that the problem of finding intermediate values is closely allied to the problem of finding a law of variation or of finding the equation of a given curve. In case a law or equation is known, unless it is a complicated one it will usually be found easier to substitute and calculate values than to interpolate them.

126. Graphic Tables.—If the variation of two quantities is known to follow the linear law it is often convenient to make a graphic table by plotting any two points and ruling a straight line through them.

Lay off a scale of values from 0 to 20 along the x-axis and label it "inches." Lay off a scale from 0 to 50 along the y-axis and label it "centimetres." Rule a straight line through the two points (0, 0) and (13, 33). Explain how a diagram of this sort can be utilized.

According to Hooke's Law, the difference in length (stretching) of a spring is proportional to the difference in force applied to it. If a spring that is hung in front of a scale has a length of 12 cm. when no weight is attached to it, and becomes 14.85 cm. long when a weight of 1 gm. is hung on it, construct a graphic table which will enable you to reduce its indicated centimetres to grams of weight (see § 104).

127. Interpolation Along a Curve.—If the change in

two variables is not in accordance with a linear law it is possible to use certain interpolation formulæ for obtaining intermediate values, but it is usually much easier to make use of graphic methods. The known data are plotted as a series of points, and these are either connected by a smooth curve or are investigated with a view to discovering an equation that will adequately represent them. If they appear to lie along a curve that has a vertical or a horizontal asymptote (Fig. 31) the hyperbola $xy = k$ may be tried, with a suitable choice of temporary axes and scales. If the curve has a single upward or downward sweep the exponential curve $y = e^{ax}$ is frequently used, but the parabola $y = ax^2$ is generally easier to handle and can usually be fitted to the given points just as satisfactorily. It may be turned so as to have its axis horizontal, if this position seems more suitable, by interchanging the variables and writing the equation $ax = y^2$.

In trying to fit a parabola to the part of the curve of $y = \log x$ that lies between $x = 5$ and $x = 15$ would you prefer to have its axis horizontal or vertical? If vertical, would its vertex be directed upward or downward? If horizontal, would its vertex be directed to the left or to the right? Plot a logarithmic curve rapidly, on a small scale, if there is any difficulty in answering the questions; compare it with the curves for $y = x^2$, $y = -x^2$, $x = y^2$, and $x = -y^2$.

128. Insufficiency of Data.—When certain tabular values are given and others are to be obtained by interpolation it must always be remembered that the known values are the only actual data and that nothing else can be obtained without making some kind of an assumption (§ 123). If it is assumed that the points all lie along a smooth curve there is always a possibility that

the assumption is incorrect. It may even happen that the given points appear to be irreconcilable with a smooth curve or with a uniform law, as in the following case: Electricity is sold by the kilowatt-hour (abbreviated KWH.) and four consumers pay the same rate to one company. The first is charged \$1.08 for 9 KWH.; the second, \$1.47 for 21 KWH.; the third, \$0.99 for 11 KWH.; and the fourth, \$1.62 for 18 KWH. Find the rate.

Plot the four points, using 1 square along the x-axis for each KWH. and 1 square along the y-axis for each \$0.10. It will be seen that it is impossible to decide where a smooth curve should run. This is a case of what is called a "step meter rate" and gives a broken line, not a smooth curve. The rate is "12c. per KWH. if less than 10 KWH. are used; 9c. per KWH. if the consumption is between 10 and 15 KWH.; 7c./KWH. if over 15." Draw the locus for this rate on the same graph and notice that it passes through the four points. The objectionable feature of sometimes charging less when the use of the current is greater is so apparent that a rate of this kind is not often used. The following is a "block meter rate" which is not so objectionable and gives approximately the same income to the company: "10c. each for the first 10 KWH. used, 7c. each for the next 5, 3c. each for all after the 15th." Plot this rate on the same diagram; also the rate, "8c. per KWH., with a minimum charge of 50c." Find a smooth curve which will come fairly close to all three of these rates.* (Sug-

* In general, a "broken line" cannot be represented, by itself, without using an equation containing an infinite series. This is the case with the last of the above rates, which is represented by the straight line $y = 50$ from $x = 0$ to $x = 6.25$ only, and the straight line $y = 8x$ for the part further to the right only.

If we are not restricted to *finite* stretches of lines it is always possible to find an equation for *both* of two loci each of which has

gestion: consider the equation $y = 12x - (1/a)x^2$ with a suitable value for a.)

129. Use of Logarithmic Paper.—A very common type of variation is that in which one of the variables is proportional to some power of the other one. For example, the distance traversed by a falling body is proportional to the square of the time of fall; the time of rotation of a planet or satellite about a particular central body is proportional to the 3/2 power of its mean distance; friction of water flowing through a pipe varies (approximately) as the 1.8 power of the velocity.. The determination, for a set of experimental data, of the proper value of the exponent is not easily accomplished by means of an ordinary graph, because the various curves $y = x^2$, $y = x^3$, $y = x^4$, etc., all have the same general shape (Fig. 37).* If logarithms are taken, however, of both sides of the equation $y = x^n$ the equivalent equation $\log y = n \log x$ is obtained, showing that $\log x$ and $\log y$

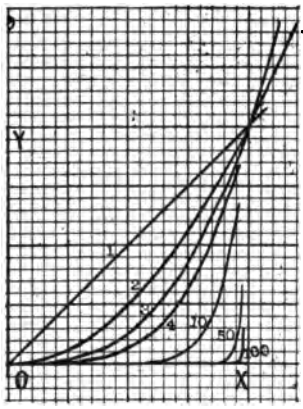


FIG. 37. LOCI OF $y = x^n$.—The curves are drawn for n = 1, 2, 3, 4, 10, 50, and 100.

its own ascertainable equation, by putting the individual equations in the form $\phi(x, y) = 0$ and multiplying them together. Thus, the *two* lines $y = 50$ and $y = 8x$ are the locus of the single equation $(y - 50) \times (y - 8x) = 0$, or $400x - 8xy - 50y + y^2 = 0$, as the student can readily show by plotting a few points or by algebraical treatment (compare § 111; 8, *d.*).

* Experimental measurements are usually of positive quantities, so that the shape of these curves to the left of the y-axis or below the x-axis is not a determining factor.

are proportional. Accordingly, if log x and log y are plotted on an ordinary graphic diagram a straight line through the origin will be obtained. Now, just as a

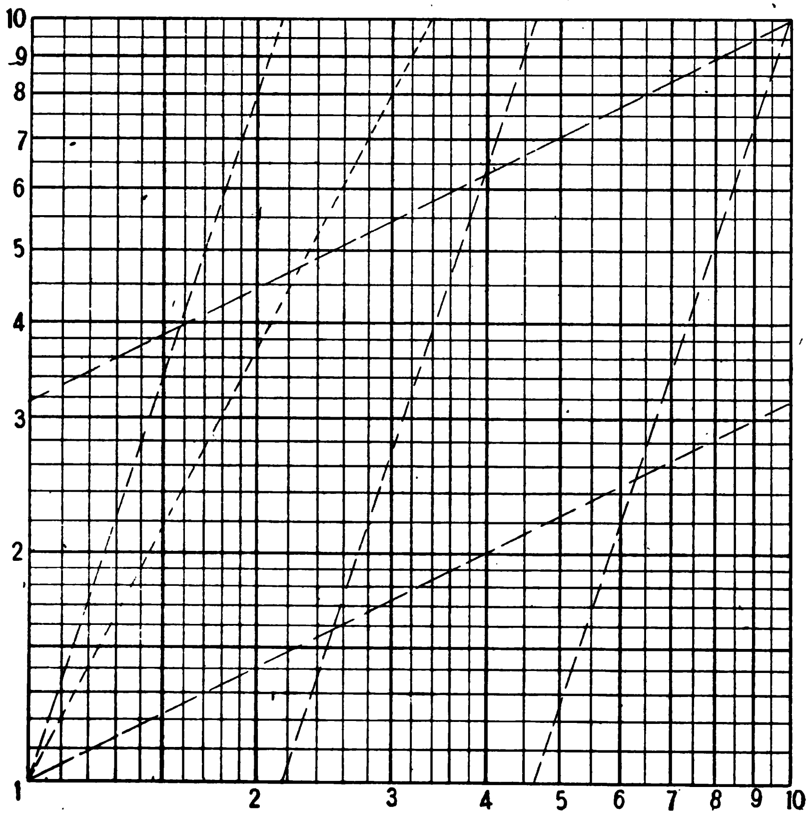


FIG. 38. LOGARITHMIC PAPER.—The logarithmic scales cause the graph of any equation of the form $y = ax^n$ to appear like $y = a + nx$ on ordinary paper, *viz.*, as a straight line.

slide rule is constructed by marking the numbers 1, 2, 3, 4, etc., at distances which are really log 1, log 2, log 3,

log 4, etc., so plotting paper can be constructed with logarithmic scales like those of the slide rule along each axis; consequently plotting x and y according to the numbered scales will really be plotting points at actual distances of log x and log y from the origin; and if these logarithms are known to be proportional it will be evident that *the graph of* $y = x^n$ *will be a straight line through the origin.* Such paper is called *logarithmic paper* and can be obtained from dealers who handle drawing materials. The lower left-hand corner of the sheet is the origin and is marked 1, 1, meaning of course log 1, log 1, or 0, 0; and the scales extend both upward and to the right from 1 to 10 as in the C and D scales of the slide rule, or sometimes from 1 to 100 as in the A and B scales. The latter arrangement would allow a single unbroken line to be used to represent the "curve" of $y = \sqrt{x}$ (Fig. 38), although of course such a line could not be drawn for the complete locus of $y = x^3$.

If a straight line is drawn on logarithmic paper without passing through the origin it must cut the y-axis at some point, such as k on the logarithmic y-scale. Then $(\log y) = (\log k) + n(\log x)$ (compare the ordinary equation of the straight line $y = a + bx$, §§ 113, 105), or $\log y = \log k + \log (x^n)$, or $\log y = \log (k \times x^n)$, or $y = kx^n$. In other words, *a straight line on logarithmic paper represents the equation* $y = ax^n$, a being the y-intercept as measured by the logarithmic scale, and n the true slope as measured by uniform scales. The diagram (Fig. 38) shows $y = \sqrt{x}$ and $y = x^3$ on logarithmic paper. Any data that are suspected of following the law $y = ax^b$ may be plotted directly on this ruled diagram and their equation determined at once.

130. Semi-Logarithmic Paper.—Consider the equation of a straight line on paper that has a uniform scale

along the x-axis and a logarithmic scale along the y-axis.

Instead of $y = a + bx$ or $\log y = \log a + b \log x$ its equation must now be $\log y = \log a + bx$. Clearing this of logarithms gives $y = a \times 10^{bx}$ or $y = a \times e^{bx}$ accord-

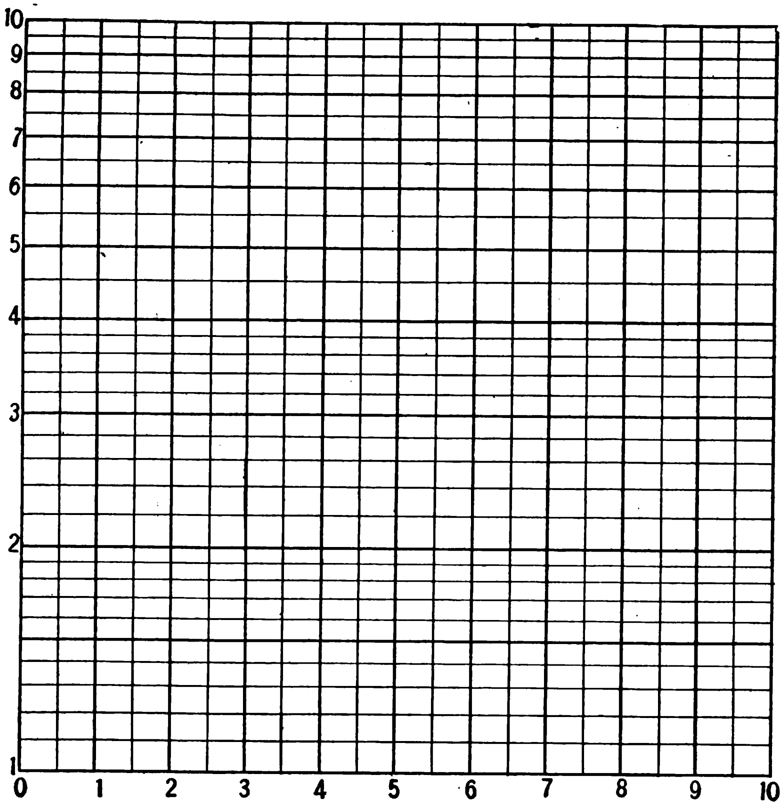


FIG. 39. SEMI-LOGARITHMIC PAPER.—The logarithmic scale along the y-axis causes a straight line through the origin to represent proportionality between x and $\log y$, *i. e.*, $\log y = kx$. The exponential law of variation, $y = ae^{mx}$, is obviously reducible to the form $\log y = kx$.

ing to the base that is used.* Consequently, *a straight line on semi-logarithmic paper (Fig. 39) represents the "exponential" type of variation,* $y = ae^{mx}$.

Draw a straight line or hold a black thread on either Fig. 38 or Fig. 39 so as to represent the area of a circle (on the vertical scale) that corresponds to the radius (as indicated on the horizontal scale). The formula is $a = \pi r^2$, or $y = \pi x^2$.

Plot the locus of $y=2^x$ on squared paper for positive integral values of x up to 6 or 7. Plot the same equation up to $x=10$ on logarithmic paper,† and also on semi-logarithmic paper. In which case is the locus a straight line? Plot $y=2^{-x}$ on semi-logarithmic paper.

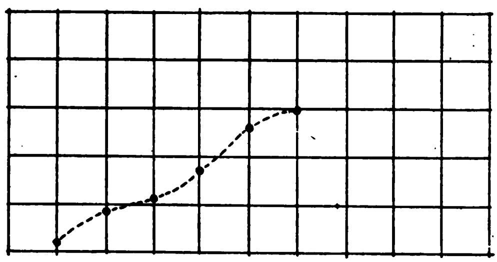


FIG. 40. EXTRAPOLATION DIAGRAM.—Graph of the relation between date and population.

Explain how to use one of these diagrams (Figs. 38 and 39) to make a graphic table of the relationship between the period (p) of vibration time of a pendulum and the length (l) of the pendulum if they are related according to the formula $p = 2\pi\sqrt{l/g}$, g being a constant.

* These two equations are of the same *form;* the first can be put into the form of the second merely by changing the *value* of b; for $10 = e^{2.30}$, so 10^{bx} is identical with $e^{2.30bx}$.

† If the specially ruled paper is not on hand the work can be done on thin paper laid over Fig. 38 and Fig. 39 of this book.

On Fig. 38 or Fig. 39 indicate the straight line that represents the equation $xy = a$ (suggestion: $y = ax^{-1}$).

131. Extrapolation.—The principles of extrapolation are like those of interpolation, but the former process is naturally more uncertain than the latter and can be trusted to give good results only when the extrapolated point is relatively near the points that correspond to the known data. As an example of the use of extrapolation the table and graph give the population of the state of California from 1850 to 1900. It is required to find the population in 1910. Continue the curve in the way that you think it would be apt to run, and note where it cuts the 1910 ordinate.

year	*population*
1850	93,000
1860	380,000
1870	560,000
1880	865,000
1890	1,208,000
1900	1,485,000
1910	

POPULATION OF CALIFORNIA

Reconstruct the extrapolated part of the curve, if necessary, so as to make it give 2.38×10^6 for 1910 and extrapolate again to determine its height for 1920. What do you find the population will be for this date?

The following example shows that extrapolation may be a very definite and decisive process if the given values follow a consistent law of change and can be carried close to the required value: Find the instantaneous velocity of a body at a certain point of time if it is known that immediately after that instant it travels 10 cm. in the first second, 7.0711 cm. in the first half second, etc., as given in the next table. Plot the tabular values with a scale of time along the x-axis and a scale of average velocity along the y-axis and extrapolate graphically to find the velocity in *no* interval of time.*

* In *no time* a moving object will of course traverse no space if its velocity is not infinite, and there is strictly speaking no meaning for such a phrase as "instantaneous velocity." It is convenient,

132. Questions and Exercises.—1. Write a definition of what you understand by the terms *interpolation* and *extrapolation*.

2. Turn to the table of values for Chauvenet's criterion (§ 208) and plot l as a function of n, using enough points to determine whether the values of l for integral values of n that are not given in the table (*e. g.*, $n = 31, n = 70$) can be satisfactorily obtained by graphic interpolation.

time (*sec.*)	*space* (*cm.*)	*velocity* (*cm./sec.*)
1	10.000	10.000
0.5	7.0711	14.142
0.3	4.5399	15.133
0.2	3.0902	15.451
0.1	1.5643	15.643
0.05	0.7846	15.692

AVERAGE VELOCITY OF A PISTON ROD THAT MOVES WITH SIMPLE HARMONIC MOTION.—Its total path is assumed to measure 20 cm.; its period, 4 seconds; the required velocity is that at the central point of its motion.

3. Turn to your graphic diagram of $y = \log x$ and draw a straight line from the point (2, log 2) to the point (3, log 3), thus making a *chord* of the curve. Mark the middle point of the chord. Is the ordinate of this point equal to the average of log 2 and log 3?

What operation can be performed on any two numbers by averaging their logarithms? Label your diagram so as to show clearly the distance that corresponds to $\log \frac{2+3}{2}$ and the distance that corresponds to $\log [(2 \times 3)^{1/2}]$. Which of the δ-formulæ for approximate calculation with small magnitudes does this diagram illustrate? In what way does it show the *equality*

however, to consider that a body which speeds up from a condition of rest to a definite velocity must have passed through all intermediate velocities successively, and as its velocity must have been continually changing the concept of a definite *velocity at a certain point* of space or time becomes almost a necessity. For purposes of rigorously logical deduction this concept is defined as a limiting value in the way indicated above.

expressed by the formula, and in what way does it show that this equality is not exact but only approximate?

4. In finding the root of an equation by the method of double position would it be satisfactory to extrapolate instead of interpolating? Explain why.

5. Calculate the roots of the equation

$$(x^2)^{(x^2)} = \pi + \log(\sin x).$$

(Suggestions: Draw a rough diagram of $y = \sin x$, say from $-\pi$ to $+3\pi$. Then add a rough outline of $y = \log(\sin x)$, remembering that $\log 0 = -\infty$, and that negative numbers have no [real] logarithms. Draw also $y = \pi$, and then $y = \pi - (x^2)^{(x^2)}$. It will now be obvious that the graph of $y = [\pi - (x^2)^{(x^2)}] + [\log(\sin x)]$ cannot cut the x-axis in more than two points. Calculate each of them separately by the method of double position. Their sum should be 1.368.)

6. A line is indicated on Fig. 38, page 151, which passes through the points (1, 1) and (3, 8) of the logarithmic paper. Determine the equation which it represents.

7. Plot the locus of $y = a + bx + k/x^2$ and find the asymptotes of the (oblique-angled) hyperbola which is obtained.

XII. COORDINATES IN THREE DIMENSIONS.

Apparatus.—A pencil with a sharp point; a "quadrangle" or "topographical sheet" of the U. S. Government contour map; (model of a small area of the map, made by piling up contour sections sawn out of thin wood).

133. Coordinates of a Point in Space.—Just as the position of a point in a plane (*i. e.*, in two-dimensional space) can be represented by two coordinates, an x-value and a y-value, giving its distance from two mutually perpendicular axes, so the location of a point in unrestricted, three-dimensional space can be fixed by three coordinates, a set of three numerical values (x, y, z) which indicate its distance from each of three planes that intersect each other at right angles. Thus, the point in Fig. 41 is at a distance of x units along the x-axis from the plane YZ of the other two axes, and at a distance of y units, parallel to the y-axis, from the plane XZ, and at a distance of z, parallel to the z-axis, from the plane XY.

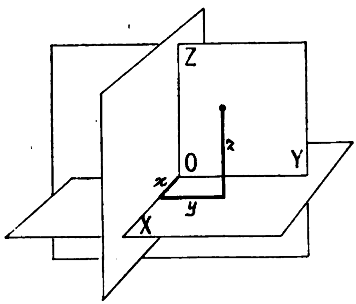


FIG. 41. COORDINATE PLANES.—These planes of reference for three-dimensional space correspond to the base lines of reference, or coordinate *axes*, that are used for two-dimensional space.

134. Convention in Regard to Signs.—The XY plane may be thought of as being represented by the same sheet

of paper as that on which the flat graphs of x and y have previously been traced (Fig. 42). Then any point whatever must be located a certain distance above or below some definite position in the plane of the paper. The x-value and y-value for this *position* will be the x and y of the *point in space*, and the distance from the point to the plane will be the z of the point. It is customary among physicists and astronomers to consider a distance above the plane of the paper as a positive value of z, and distance below it as negative, according to the arrangement of axes shown in Figs. 41 and 42. The

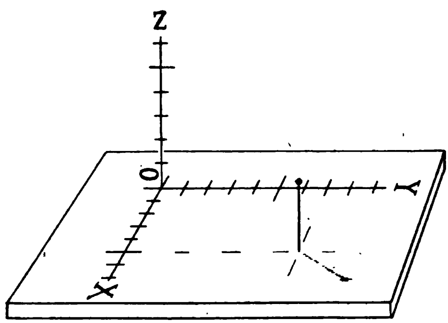


FIG. 42. POSITIVE DIRECTIONS OF AXES.—Z is considered positive when measured upward from the normal plane of the x-axis and y-axis. The pin shown in this diagram would have the location of its head represented approximately by the position (5, 7, 3), the numbers in parenthesis being used for x, y, and z, in order.

convention among pure mathematicians is just the opposite, *viz.*, distances above the paper are called negative and those below are positive (Fig. 43). The distinction is immaterial in the greater part of the study of pure mathematics, but it is important in many branches of applied science; for example, the equations of the curve of a right-hand screw thread in one system will represent

a reversed or left-hand thread if the other system is used instead. Accordingly, there is a tendency among pure mathematicians to use the "physical" arrangement for the sake of uniformity, and it is the only one that will be used in this book.

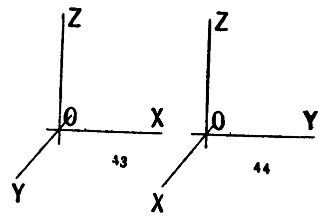


FIG. 43. FIG. 44.

CONVENTIONS AS TO SIGNS.—Fig. 43 corresponds to the arrangement that is understood in pure mathematics; Fig. 44 is that of applied mathematics.

That these two arrangements exhaust the possibilities may be seen by taking any arbitrary arrangement of axes at right angles to each other and rotating them as a rigid figure. They can always be made to coincide with either Fig. 43, or Fig. 44. The two are essentially different because neither one can be rotated in three-dimensional space so as to coincide with the other.*

In each of the following exercises the positive directions of the axes are indicated. Consider each in turn and state whether it is like Fig. 44 ("right") or the opposite ("wrong").

* Notice that a plane xy-diagram drawn with $+ Y$ upward but with $+ X$ to the left cannot be rotated in its own two-dimensional space so as to coincide with the conventional arrangement. It can be turned through the third dimension, however (turning the upper surface of the paper downward), and made to coincide. Similarly, a solid model of Fig. 43 would need to be turned through a fourth dimension of space before it could be made to coincide with a solid model of Fig. 44. Since we have no appreciation of a fourth dimension the two figures are as essentially different to us as a capital L and a Greek capital Γ would seem to a being whose sense perceptions were limited to space of two dimensions.

1. X points northward; Y, upward; Z, westward. (Ans.: wrong.)
2. X, west; Y, up; Z, north. (Ans.: right.)
3. X, down; Y, east; Z, south.
4. X, down; Y, north; Z, east.
5. X, down; Y, west; Z, north.
6. X, down; Y, south; Z, east.
7. X, east; Y, north; Z, up.
8. X, west; Y, north; Z, up.
9. X, west; Y, north; Z, down.
10. X, up; Y, east; Z, north.

135. Loci of Simple Three-Dimensional Equations.— In studying geometrical relationships in a plane we have seen (§ 105) that the equation $y = a$ represents all the points that are a units above the x-axis, that is, a straight line parallel to the x-axis and situated at a distance a above it. In the same way, the equation $z = m$ must denote all points located m units above the xy-plane, *e. g.*, the points $(2, 3, m)$, $(0, 0, m)$, $(0, 10, m)$, etc., since each of these groups of values ($x = 2$, $y = 3$, $z = m$, for example) will satisfy the equation. Similarly, $x = m$ or $y = m$ will denote a particular plane parallel to YOZ (Figs. 41 to 44) or to XOZ, and so perpendicular to the x-axis or to the y-axis, respectively.

Consider next an equation that contains only two of the three variables. On a plane surface $y = x^2$ is a curve, a parabola. In space, any point above or below any point of the curve will satisfy this equation, for it has the same x and y, but a different z. Accordingly, the equation represents the surface that is made up of all the vertical lines that can be passed through points that lie on the plane curve.

If the equation contains all three variables, any arbitrary value may be assigned to x, any value whatever to

y, and the value of z will then be determinate. That is, the points of the locus will be situated at varying distances above *all* the points in the xy-plane, and so will comprise a surface.

In general, then, *a single equation denotes a surface.* Since two surfaces intersect along some line *two simultaneous equations will denote a curve in space.* It has been seen (§ 108) that $x^2 + y^2 = 5^2$ must be the equation of a circle; in the same way $x^2 + y^2 + z^2 = 5^2$ is the equation of a spherical surface that is everywhere 5 units distant from the origin. The simultaneous equations

$$\begin{cases} x^2 + y^2 + z^2 = 25 \\ \qquad\qquad z = 3 \end{cases}$$

will have for their locus the points which satisfy the first equation and at the same time satisfy the second; *i. e.*, each of the points that is located on the spherical surface $x^2 + y^2 + z^2 = 25$ and is at the same time in the horizontal plane $z = 3$. Such points must lie on the intersection of the plane and the spherical surface; and this intersection is known to be a curve, namely, a circle. For other values of z the circular intersection would be larger or smaller; for the (tangent) plane $z = 5$ the intersection would shrink to a point, for $z = 6$ there would be no intersection.

136. Contour Lines.—If a were given all possible values in the equation $z = a$ the resultant loci would be horizontal planes at all levels, and they would intersect the sphere $x^2 + y^2 + z^2 = 5^2$ in all the horizontal circles that could be drawn on its surface. All of these intersections, consequently, *would be* the spherical surface; a smaller number of them would form a sort of skeleton, from which the shape of the surface could be inferred

if they were sufficiently numerous. An extremely useful way of representing a surface, especially an *irregular* surface, on paper is by outlining the intersections which would be formed by horizontal planes at various levels. The diagram (Fig. 45) shows a portion of the surface

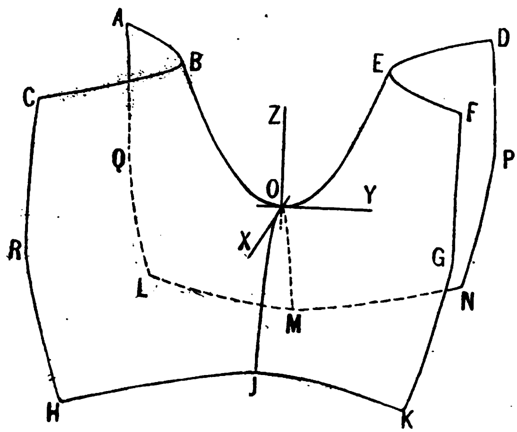


FIG. 45. THE HYPERBOLIC PARABOLOID $x^2 - y^2 + z = 0$.—A surface of this general character is sometimes spoken of as a "saddle-back."

$x^2 - y^2 + z = 0$, called a *hyperbolic paraboloid.* When x is zero the equation reduces to $z = y^2$; *i. e.*, the intersection of the curve with the YZ-plane is the parabola $z = y^2$, BOE in the diagram. When $y = 0$ the section MOJ is a parabola $z = -x^2$. At any level where $z = a$ the equation of the surface becomes $y^2 - x^2 = a$, a hyperbola such as ABC and DEF or HJK and LMN; for $a = 0$ this degenerates into the two straight lines that are common asymptotes, $y = \pm x$. A series of horizontal sections of the surface are shown in Fig. 46 as they would appear if they were all viewed from above,

the algebraical signs showing whether each curve is above or below the xy-plane and the smaller numbers denoting the lower levels while the higher numbers indicate the upper ones. If the numbered lines are imagined to be raised 1, 2, 3, 4, 5, 6, 7, 8, and 9 centimetres respectively above the surface of the paper it will be evident that a good idea of the shape of the curved surface which they outline can be obtained without the necessity of consulting a perspective drawing like Fig. 45. Such a surface as this one (which is convex upward along the x-axis and concave upward along the y-axis) is commonly called a *saddle-back* and represents roughly the shape of the surface of the earth in a *mountain pass.*

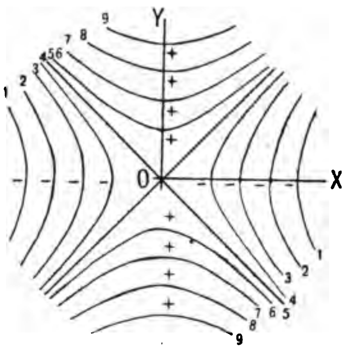


FIG. 46. CONTOUR LINES OF A SURFACE.—Horizontal sections at different levels of the curved surface of Fig. 45.

The irregular surface of the earth is sometimes represented on maps by horizontal section lines (*contour lines*) which make it easy to find the location, height, slope, etc., of any hill, valley, or other surface character by inspection of the map.

137. Use of Contour Maps.—The lines of horizontal section usually correspond to heights taken at equidistant intervals, for example, at 20, 40, 60, 80, . . . feet above mean sea level, and are called *contour lines.* Students usually find it most convenient to think of them as representing the new *shore lines* that would be formed if the

sea level were to rise 20 ft., 40 ft., etc., above its original level. The uniform interval, twenty feet in this example, is called the *contour interval,* and the level of reference,

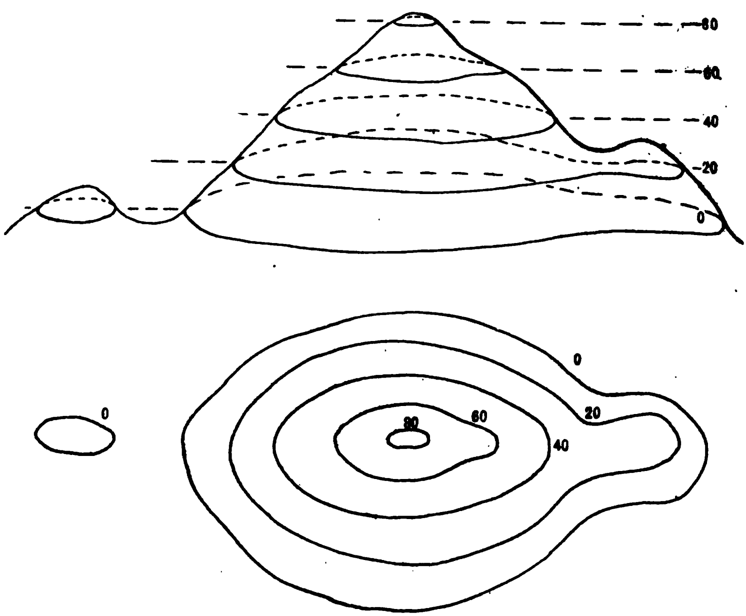


FIG. 47. CONTOUR LINES OF A HILL.—The upper figure is a side view of a hill 80 feet high, showing in perspective the outlines that would be produced if it could be cut into twenty-foot slices or the new shore lines that would be produced if the surrounding country could be flooded to depths of 20, 40, 60, and 80 feet.

The lower figure is a set of *contour lines* that represent a top view of the levels shown above. The usefulness of a map is greatly increased by having contour lines drawn or printed on it in some distinctive color. Such maps usually have the contours in brown, rivers and lakes in blue, roads, buildings, boundaries, etc., in black.

The line 20 is the locus of all points where the surface is 20 ft. above the plane of reference. Notice that the ground is always higher on one side of a contour line and lower on the other.

usually mean sea level, is called the *datum plane.* The diagrams (Fig. 47) show a vertical section of a hill 80 feet high and a horizontal plan of its contour lines.

Examine the contour map and notice whether the datum plane and the contour interval are stated in the margin.

Find a hill or other elevated area on the map, and notice the arrangement of the contour lines. What is the difference, as indicated on the map, between a high hill and a low hill? What is the difference between a steep hill-side and a more gradual slope?

The model shows the 100-foot contour lines of a hill that is given on the map. See if you can identify it from the shape of the horizontal sections.

Find a brook that runs down a hill. In what general *direction* do the contour lines cross the brook? Why? What is the general *shape* of the contour lines where there is a water-course? Where a hill sends out a projecting buttress or ridge what is the general contour form? How is a plateau formation indicated by contour lines?

Why is it that contour lines where they run across a road-way are never as near together as they often are in other localities? Find a road (on the map) that appears to have been purposely so constructed as to cut the contour lines at considerable intervals of distance.

138. Construction of a Contour Map.—Copy the following table of altitudes upon the squared paper of your notebook, writing each number with ink in very small figures directly on the intersection of two of the ruled lines. Make all the spaces between columns of numbers equally wide (say 3 or 4 squares), and space the horizontal lines of numbers at the same distances as the intervals between the columns. Omit one or two lines

at the bottom of the table or one or two columns at the right-hand side rather than crowd the numbers close together, if the page of your notebook is not large.

	C'hill St.	Vine	Race	Arch	Market	Chestnut	Walnut	Locust	Spruce	Pine	Lombard
F	20	15	25	22	26	24	16	7	8	21	29
2	27	31	32	31	31	27	14	13	16	21	27
3	27	32	34	31	27	20	23	23	24	24	29
4	30	35	36	26	20	24	23	27	26	26	28
5	32	36	37	29	23	28	20	22	30	31	29
6	32	31	31	33		28	22	26	25	31	33
7	35	36	34	38	29	27	28	26	27	32	35
8	34	40	36	36	32	29	29	32	30	23	36
9	36	36	38	40	37	34	30	30	31	35	35
10	33	31	36	40	37	33	32	33	34	33	32
11	33	34	49	42	39	39	34	35	35	35	33
12	38	37	43	40	41	41	38	35	32	32	30
13	34	40	42	37	43	44	41	39	36	32	29
BD	46	40	39	35	42	47	45	42	39	35	31
15	44	37	37	32	41	45	42	39	37	33	35
16	38	33	35	38	38	43	43	40	37	33	35

ALTITUDES IN CENTRAL PHILADELPHIA.—The data are expressed in feet above mean sea level. No datum is given here for the corner of Sixth and Market streets.

Copy also the dotted line that represents a 34-ft. contour, noticing that it passes exactly through the points marked 34, runs half-way between the points 33 and 35, runs between 32 and 35 twice as far from the former as from the latter, etc.

Let your contour interval be 3 feet, and start a 31-foot contour line somewhere near the center of the diagram. Extend it *carefully* in both directions, remembering that all the altitudes close to one side of it must be greater than 31 and all those near the other side must be less than 31. In case it is difficult or impossible to extend the line further than a certain point leave it and begin the construction of another contour line. After the easier lines have been finished they will be found of considerable aid in helping to determine the course of the more difficult ones. Remember that a single level may be represented by two or more lines that do not join; thus, there will be a 34-foot ring around the 37-foot peak at Fifth and Race Streets, and this cannot connect with the line shown on the diagram because of the low-lying ground between. If the sea level were raised 34 feet the line indicated here would be the new shore line and the 34-foot ring would be the shore of a separate island.

Draw new contour lines at intervals of every three feet until the whole area is covered.

Estimate the elevation at the corner of Sixth and Market Streets by interpolation along Market Street (E. and W.), also by interpolation along Sixth Street (N. and S.) and along two diagonal lines (NE. and SW., and NW. and SE.). Finally decide for yourself the most reasonable value for this elevation.

139. Questions and Exercises.—1. If $x/a + y/b + z/c = 1$ is known to represent a plane surface what can

you *prove* about the position of this plane (compare § 116)?

2. Write the equation of the cone (curved surface) produced by rotating the straight line $z = 3x$ around the z-axis (suggestion: rotation changes each point that was located at a distance of x to the right of the z-axis into a horizontal circle whose radius is equal to that x-value).

3. What kind of a locus corresponds in general to three simultaneous equations?

4. Can two contour lines representing different levels ever intersect each other on a map? What would be true of the earth's surface at the point of intersection?

5. Show that the curves *RCAQ* and *GFDP* made by cutting the hyperbolic paraboloid (Fig. 45) at any distance to the left or the right of the origin by the plane $y = a$ are "arch-shaped" parabolas. Show that the curves *RHKG* and *QLNP* made by limiting the surface at the front or back by the plane $x = a$ are "festoon-shaped" parabolas.

XIII. ACCURACY

Apparatus.—Rectangular wooden block measuring about $4 \times 8 \times 8$ cm.; centimetre and millimetre scale; one scale (of centimetres and millimetres) and one steel tape-measure to be used by the whole class.

140. Significant Figures.—Significant figures have already been defined as all of those that compose a number except one or more ciphers at the extreme left or right which may be necessary to express the order of magnitude of the number (*i. e.*, to locate the proper position of the decimal point) but are not needed in any other way for indicating its value. For example, .003803, 3.803, and 3803000 have four significant figures each; provided that the last one is a "round number" (*i. e.*, is meant to be accurate only to a whole number of thousands). The ambiguity in the last case can best be avoided by writing 3.803×10^6; and the same three numerical values could all be written with seven-figure accuracy by expressing them as 3.803000×10^{-3}, 3.803000, and 3.803000×10^6 respectively.

It is usually understood, in any kind of careful scientific work, that a number is never written with too many significant figures, which would appear to give it an unwarranted degree of accuracy; nor with too few significant figures, which would mean a neglect of the full extent of the accuracy that had been obtained. In other words, figures are known to be correct as far as they are stated and are unknown for all notational "places" beyond. For example, if 1 yard is found to be 83.8 cm., the best value of 1 foot that can be calculated from this determination is 1 ft. = 27.9 cm. and

the ordinary arithmetical result, 1 ft. = 27.93333 cm., is not only unjustified, but in this case is positively wrong although the statement that 1 yard = 83.8 inches is right. cm.

Which four of the following values for e are correct and which six are incorrect: 2.71828182; 2.71828183; 2.718281; 2.7182; 2.7183; 2.7180; 2.7; 2.70; 2.71; 2.72?

141. Infinite Accuracy.—A number like the ratio of the diagonal of a square to its side or the ratio of circumference to diameter for any circle, which depends only upon theoretical considerations, can be calculated with any desired degree of accuracy, but no number whose value depends upon the measurement of a material thing can be stated with more than a definite degree of accuracy, on account of the limitations of the method of measurement. If an imaginary circle is made large enough to enclose the Milky Way its circumference, measured in terms of its radius, must be equal to $2 \times 3.14159265358979323846264338327950288419716 94$, but if it were possible to measure a real line of corresponding dimensions the best microscope in existence would not enable us to decide whether the value of π should be as small as 3.1415926535897932384626433832 79 or as large as 3.1415926535897932384626433832 80.* The numerical value of π has been calculated as far as 707 decimal places, but this of course is not even an approach to perfect (*i. e.*, infinite) accuracy.

142. Relative Errors.—If the thickness of a lead pencil is measured by holding it in front of a scale the separate

* Such a circle might have a circumference of 100,000 "light-years," or, say, 10^{23} centimetres, considering the velocity of light to be 30,000,000,000 cm. per second and the number of seconds in a year to be over 30,000,000. If a "homogeneous immersion" objective can separate two points at a distance of 1 or 2×10^{-5} cm. it could measure with an accuracy of nearly 10^{-28}, corresponding to 28 significant figures.

measurements made by a class of students will be likely to vary as much as 0.03 cm. This is nearly four per cent of the distance measured. If the length of a six-foot table top is determined with an ordinary steel tape measure the results that are obtained will be apt to vary two or three millimetres. This is about $\frac{1}{6}$ per cent. of the distance measured. In the latter case the error is about ten times as great, considered as an isolated length, as it is in the former. Considered in relation to the thing measured, however, 0.16 per cent is much smaller than 4 per cent. Since the measurement of the table is obviously a more accurate process than that of the lead pencil it will be evident that accuracy is a matter of relative size, not of absolute size. To state that a measurement is uncertain by 4 per cent means something definite, but the statement that a measurement is correct "within 0.03 cm." tells us nothing about its accuracy, if nothing is stated about the length measured.

Measure a lead pencil and a table in the manner described. The other members of the class are to measure the same objects with the same ruler and the same tape. Report the measurements to the instructor to be tabulated. Then determine their maximum discrepancy, both in centimetres and as two percentages.

By using the most refined methods it is possible to measure a distance of several miles with remarkable accuracy. If an accurate steel tape is used, which is stretched by a measured force when it is at a carefully determined temperature, it is possible in the course of a few weeks to measure a base line for surveying purposes with an error of about one unit out of 500000. Make a rough mental calculation (assume 5000 ft. = 1 mile) of what such an error would amount to in measuring a

distance of ten miles. Is it larger or smaller than the 0.03 cm. of the lead-pencil measurement?

Does the accuracy with which a number is stated depend upon the number of decimal places to which it is carried out, or upon the number of significant figures which it contains?

143. Uncertain Figures.—The figure that follows the last trustworthy figure of a measurement may be uncertain to the extent of two or three units, and yet its approximate value may be definite enough to make one hesitate to discard it. This would probably be the case if the student attempted to estimate *hundredths* when using a scale of whole centimetres without millimetre graduations. In such cases it is better to retain the doubtful figure, keeping in mind however the fact that the result obtained from it in any calculation will be liable to have one uncertain figure also.

Another case in which it is sometimes desirable to keep a single uncertain figure is when a higher degree of accuracy is made possible by retaining it during the process of a calculation, although it is eventually discarded in stating the final result. This is illustrated in the treatment of the "figure last canceled" in the processes of abridged multiplication and division.

144. Superfluous Accuracy.—An appreciation of the degree of accuracy that is required in particular cases often makes it possible to avoid needless trouble in measuring or calculating. If the dimensions of a rectangular block cannot be measured with more than three-figure accuracy its density can be obtained by weighing it merely with three-figure accuracy, and the determination of the fourth figure of its mass or weight will not enable a better calculation of its density to be made.

Obtain the dimensions of a rectangular wooden block

as accurately as possible with a scale of centimetres and millimetres, estimating tenths of a millimetre. Calculate its volume, using the proper number of significant figures, and then find its density by weighing it and dividing mass by volume.

145. Finer Degrees of Accuracy.—A measurement may happen to be known with an accuracy greater than can be expressed by three significant figures and yet not with four-figure accuracy; that is, the step from any degree of accuracy to a ten-fold greater degree may be too great to be suitable in all cases. For example, a period of time equal to $125\frac{3}{5}$ seconds may have been measured more accurately than to the nearest whole number of seconds and yet may not have been determined closely enough to enable tenths of a second to be stated. In such cases common fractions may be used instead of decimals; and a significant zero can be employed in such a form as "$125\frac{0}{5}$ seconds" without any lack of intelligibility, meaning of course "between 124.9 and 125.1 seconds," or "nearer to 125 seconds than to $124\frac{4}{5}$ or to $125\frac{1}{5}$." It is usually more satisfactory, however, to state a numerical value for a measurement and then affix a statement of its uncertainty in the form of a percentage or otherwise.

What would you understand to be the largest possible error in a measurement stated to be 125 seconds? Ans.: 0.5 sec., or 0.4 per cent.

What would you understand to be the largest possible error in a measurement stated to be 125.0 seconds? What would it be for $125\frac{0}{5}$ sec.? Express the answers both in seconds and in percentages.

Consider carefully the stated measurement "$22\frac{1}{3}$ cubic centimetres." If it had to be written as a decimal for the sake of averaging it with a set of other measurements

would you prefer to write it 22.3 or 22.33, or would you round it off to 22? Explain why.

146. Possible Error of a Measurement.—Instead of stating that a measurement may have an error in a certain decimal place or significant figure although the figures that precede it are correct it is usually better to state the possible error in the form of a ratio or a percentage. Since the figures of a numerical statement are intended to be significant and correct as far as they go it is evident that the statement cannot have an error larger than half of a single unit in the last decimal place, or five units in the place that would follow the last one that is written. Accordingly, a statement, like that of § 56, that a length which is written 174.2 certainly cannot have an error of more than 1 out of 1742, is on the safe side but could be improved by making it read "not more than $\frac{1}{2}$ out of 1742, or 1 out of 3484, or .0003, or .03 per cent."

Turn to your experimental determinations of sines (§ 40), find the greatest possible error of each as half of a single unit in the last written decimal place, and reduce this possible error either to a decimal fraction (*e. g.*, 1 out of 25 is the same as .04) or to a percentage (1 out of 25 = 4 per cent), according to which form appeals to you as being the more expressive. Then mark each error "small," "large," etc., as given in the table of errors classified according to size (see appendix).

Notice that a gradation which is as coarse as that expressed by enumerating significant figures is unsatisfactory in certain cases. The number .9624 has no more significant figures than 1.093 but is expressed with about ten times as great accuracy as the latter. For purposes of using it in a calculation along with 1.093 it should be rounded off to .962. According to the rules

given in the chapter on small magnitudes $1.093 \times .9624$ would equal $1. + .093 - (1 - .9624) = 1. + .093 - .0376 = 1.0554$. Explain why this result is unjustifiable, and show the fallacy in the process of obtaining it.

147. Possible Error After a Calculation.—*The possible error of the* **sum** *of two or more measurements is equal to the sum of their individual possible errors.*

A table is found by measurement to be 44.3 cm. higher than a bench which has a height of 42.5 cm. from the floor. If the third significant figure of each measurement may have a possible error of half a unit what limits can be assigned for the height of the table from the floor. Ans.: between 86.7 cm. and 86.9 cm.

The possible error of the **difference** *between two measurements is equal to the sum (not the difference) of their possible errors.*

A table is 51.1 cm. lower than a shelf which is between 137.9 cm. and 140.1 cm. above the floor. How high
9 must the table be? Ans.: from 88.75 cm. to 89.05 cm.

Would it be correct to answer the last example, "from 88.8 cm. to 89.0 cm."?

The possible error of the **product** *of the two measurements* $m_1 \pm e_1$ *and* $m_2 \pm e_2$ *will be found by ordinary multiplication to be* $m_1e_2 + m_2e_1$ under ordinary circumstances in which the error is small (§ 70) compared with the measurements themselves, so that e_1e_2 is negligible (§ 72). It will be instructive, however, to re-write the measurement and its possible error, $m \pm e$, in the form $m(1 \pm e/m)$ before performing the multiplication. The fraction e/m will now represent the relative error (§ 54). The product will be $m_1(1 \pm e_1/m_1) \times m_2(1 \pm e_2/m_2) = m_1m_2(1 \pm e_1/m_1 \pm e_2/m_2)$; *i. e., the possible percentage error of a* **product** *is equal to the sum of the possible percentage errors of its factors.*

A rectangular surface measures 50 cm. $\times$ 20 cm. Calculate $(50 \pm 0.5) \times (20 \pm 0.5)$, and show that the possible error of the area is 35 cm²; then find the relative value of 0.5 to 50 and that of 0.5 to 20, and see whether their sum is the same as the ratio of 35 cm² to 50 cm. $\times$ 20 cm.

Similarly, *the possible percentage error of a* **quotient** *is equal to the sum of the possible percentage errors of the divisor and dividend;* for $m_1(1 \pm e_1/m_1) \div m_2(1 \pm e_2/m_2)$ is equal to $(1 \pm e_1/m_1 \pm e_2/m_2)m_1/m_2$.

These rules are equally applicable in cases where one of the two quantities employed has no error whatever. If a 50-foot tape measure has a possible error of a tenth of a foot (0.2 per cent) then there may be an error of three tenths of a foot in using it to measure a distance of 150 feet. If the measured diameter of a circle is uncertain by 3 per cent then the calculated circumference will be liable to an error of 3 per cent.

148. " Probable " Error.—The magnitude of a "possible error" is not actually as definite as it is usually assumed to be. There may be a high degree of probability that a measurement has no error greater than half a unit in the last decimal place that is written, but this can never be so high as to amount to an absolute certainty.

The most that can be said is that a certain large range for the error of a measurement is very improbable, narrower limits of error are less improbable, that an error is within a still smaller range may be probable rather than improbable, that the error has at least a certain minute magnitude may be so highly probable as to be almost a certainty, etc. Without taking up a quantitative discussion of the relative likelihood that the magnitude of a particular error will fall within given

limits we need notice here only that the most convenient degree of probability that we can specify is that of a half-way point between probability and improbability. *For a certain limited class of error* it is possible to state that a particular error is "*just as likely as not*" to be within certain numerical boundaries, *i. e.*, that the error's chance of being smaller than the stated limit is just equal to its chance of being larger than the limiting value. Boundaries of this sort are sometimes called *probable errors*, and will be considered in a later chapter. At present all that needs to be noticed in regard to them is that they are not "probable" in the sense of being highly probable values nor are they even "errors" except in a restricted sense of the word, so that the accepted name is apt to prove a misleading one.

149. Accuracy Required in Special Cases.—When only a certain degree of accuracy is needed there is nothing to be gained by attempting a greater degree, and there is usually a considerable loss of time and labor if calculations are carried out to more significant figures than can be utilized. For a great many scientific, engineering, and commercial calculations three-figure accuracy is sufficient, and in such cases the slide rule can be used more readily than a table of logarithms. Four-figure accuracy is attained by the use of four-place logarithm tables, and is sufficient for most of the calculations of physics and chemistry, although five-figure accuracy is needed for certain measurements, *e. g.*, determinations of mass, periods of time, barometric pressures, etc. Six-figure accuracy is sufficient for all surveying except the most precise geodetic measurements. Seven-figure accuracy is also needed for certain astronomical calculations.

150. Questions and Exercises.—1. An expression like

"four-and-a-half-figure accuracy" is sometimes seen. Write down your opinion of what this phrase means, expressed as a percentage; and state your reasons for choosing this number.

2. Explain why the possible error of a difference is equal to the sum (instead of difference) of the errors of its components.

3. If a wooden block measures 250 cm^3, with an uncertainty of 5 cm^3, and weighs 2.00×10^2 gm., how many gm./cm^3 does the uncertainty of its density amount to? Ans.: sixteen or eighteen hundredths of a unit. What can you say of the effect of the uncertainty of the weight?

4. Each volume of a ten-volume encyclopedia has 2.0 inches thickness of leaves between two covers that are each 0.2 inch thick. If the number of pages per volume is liable to vary by 2 per cent and the thickness of the cover by 1 per cent, what is the maximum length of shelf-space that may be needed for the set? Ans.: 24.44 inches.

If a calculated measurement M is equal to $10m_1 + 20m_2$ what will its possible error E amount to in terms of the possible errors e_1 and e_2 of m_1 and m_2 respectively? Suggestion:

$$\frac{e_1}{m_1} + \frac{0}{10} = \frac{E_1}{M_1}, \qquad \text{where} \quad M = M_1 + M_2 \ (\S\ 147).$$

5. The quotient in § 147 should, strictly speaking, be $(1 \pm e_1/m_1 \mp e_2/m_2)m_1/m_2$. Explain why the inversion of the double sign is unnecessary in this connection.

6. Write in your notebook a brief *résumé* of all of the important points in chapters iv and xiii.

XIV. THE PRINCIPLE OF COINCIDENCE

Apparatus.—Two metre sticks; cardboard; knife; rectangular wooden block; slide rule.

151. Effect of Magnitude upon Accuracy.—With a scale of millimetres measure a length of one inch as accurately as possible, writing down the number of whole centimetres, additional whole millimetres, and estimated additional tenths of a millimetre which it contains. Measure a length of two inches in the same way and divide the result by two. Calculate one tenth of the carefully measured length of an interval of ten inches and compare with the two other results. Notice that a large quantity can usually be measured more accurately than a small one. If the method of measurement gives the same absolute accuracy in two different cases (for example, no error greater than .005 cm.) the ratio of error to measurement will naturally be a smaller relative error just in proportion as the measurement itself is greater. It is for this reason that the metre was chosen for a standard of length (§ 19) instead of the centimetre, and the kilogram for a standard of mass instead of the unit, a gram.

152. Measurement by Estimation.—In order to make a more accurate determination of the length of an inch lay one metre stick on the table with the metric graduations upward and place the other beside it with the graduation in inches and tenths of an inch upward. See that the scales are in close contact and note down the number of centimetres and hundredths that happens to be opposite the mark 1 inch on the other scale. Note also the indication of the mark 31 inches. Move one

scale along the other at random and repeat the observations, unless otherwise directed, until ten sets have been taken. The distance measured should always be the same, 30 inches, but it should be measured at other places, such as the mark 4 and the mark 34. Never use the mark 0 at the very end of the stick, as it is often inaccurate on account of wear.

Tabulate the results in a *form* like the following:

MEASUREMENT OF AN INCH BY THE METHOD OF ESTIMATION.

	1st line	*2d line*	*Interval*	
Cm. scale....	20.05	45.50	25.45	
Inch scale....	1.00	11.00		10.00
Cm..........	50.00	75.47	25.47	
Inch........	2.70	12.70		10.00
Cm..........	10.06	35.46	25.40	
Inch........	1.50	11.50		10.00
.	.	.	.	
.	.	.	.	
.	.	.	.	
Sum........			254.02	100.00
Average........			25.402	10.000

Length of an inch: — 2.5402 cm.

153. Measurement by Coincidence.—In the method of measurement *by estimation* one or more equal intervals are measured in the customary manner, tenths of the smallest scale-divisions being determined by a mental estimate. For the method of measurement *by coincidence* a series of equal quantities is necessary. If these can be directly compared with a series of equal scale-units it will usually be possible to observe that some whole

number of the equal quantities is of precisely the same magnitude as some other whole number of scale-divisions. For example, the diagram (Fig. 48) shows in two different

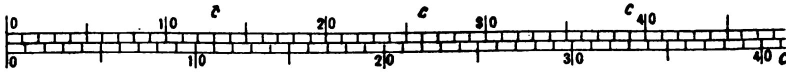


FIG. 48. THE METHOD OF COINCIDENCES.—Each of the unknown intervals is seen to be a trifle less than 1 cm., but the fact that there is no distinguishable difference between 13 intervals and 11 cm. shows that the length of the unknown interval must be 11/13 of a centimetre.

locations (*c*) that 13 of the unknown intervals are equal to 11 of the centimetre units. Accordingly, the unknown interval must be 11/13 cm.

Set the two scales so that any whole number of inches near one end of one metre stick is exactly opposite some whole number of centimetres on the other. Hold the two sticks firmly together and look again to see that the coincidence is exact. Do not be satisfied unless you are unable to say whether the upper mark is a little toward the right or toward the left of the one below it. Without allowing the two scales to slip find another place, at least 30 cm. distant from this point and preferably further, where there is another exact coincidence, this time between any centimetre or millimetre graduation and any graduation of inches or tenths. Try to decide whether the coincidence is exact or whether the imaginary central axis of one line lies a little beyond the other, and if the coincidence is faulty choose a better one elsewhere.

Record the results as in the specimen table, re-set the two scales, and repeat until 10 determinations have been made. Shift the position of the first coincidence, as was done in the method of estimation; also, take care

that the interval between coincidences is not of the same length each time.

MEASUREMENT OF AN INCH BY THE METHOD OF COINCIDENCES.

	1st line	*2d line*	*interval*	*cm. in 1 in.*
Cm.......	90.000	35.400	54.600	
In........	4.000	25.500	21.500	2.5395
Cm.......	15.000	71.400	56.400	
In........	6.000	28.200	22.200	2.5405
⋮	⋮	⋮	⋮	⋮
Average..				2.5401

It is not absolutely accurate to assume that the coincidences are always exact to a hundredth of the smallest graduation of the scale, but the lack of perfect coincidence is usually perceptible even if the discrepancy is much less than a tenth of a scale-division. It will very rarely exceed a hundredth of a smallest subdivision if the work is carefully done.

154. The Vernier.—Rule a straight line along a strip of cardboard 5 or 10 cm. wide and about 20 cm. long (Fig. 49). As shown in the diagram, lay off a scale of

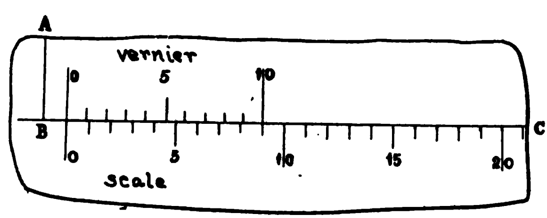


FIG. 49. MODEL OF A VERNIER CALIPER.—The diagram can be drawn on cardboard, cut out, and used for fairly accurate measurements.

centimetres along the lower side of the line; along the upper side lay off a series of ten intervals, making each one 0.9 cm. long, so that their combined length will be just 9 cm. Rule a perpendicular line, *AB*, a little to the left of the common zero point, and mark the words "scale" and "vernier" as indicated. See that your work corresponds in every respect to Fig. 49, except that the letters *A*, *B*, and *C* are not marked, and the scale of centimetres may be longer or shorter than shown here. Observe that the mark 1 on the vernier is one tenth of a centimetre to the left of the mark 1 on the centimetre scale, 2 on the vernier is two tenths to the left of 2 on the scale, etc. Put the card on a larger piece of cardboard, to avoid marring the table, and cut from *C* to *B*, but no further, also from *A* to *B*. The result will be a cardboard model of a *vernier caliper*. Notice that the zero of the vernier acts as a pointer that indicates 0 cm. when the jaws (*AB*) of the caliper are no distance apart; consequently it must give the correct reading on the centimetre scale when the jaws are slid apart to any given distance.

Holding the body of the vernier caliper stationary on the table move the vernier slightly to the right so that its line number 1 coincides with the 1-centimetre mark; then move it further until 2 and 2 coincide; then 3 and 3; etc. When the tenth division of the vernier has been brought into coincidence with a line on the centimetre scale it will be found that the zero of the vernier stands at the mark 1 cm. and the jaws of the caliper are separated by a distance of a single centimetre. From the construction of the apparatus it will be clear that when any division of the vernier (say number 3) coincides with any division of the lower scale the jaws of the caliper must be separated by a corresponding number (3) of

tenths of a centimetre more than some whole number of centimetres.

Separate the jaws of this model vernier caliper by a distance of eight and a half centimetres, as nearly as you can by estimation; then look at the vernier and notice that while its zero is beyond the mark 8 of the centimetre scale the division 5 of the vernier coincides with some division (no matter which one) of the main scale. The reading is accordingly 8.5 cm. In the same way read the indicated length when the caliper is set at random, and repeat the process until you are perfectly familiar with it.

155. Use of the Vernier Caliper.—Hold the two parts of the caliper in alignment by pinching the line *BC* between the fore-finger and thumb of the left hand, holding the lower part tightly but allowing the upper part to slide; at the same time hold the same line between the fore-finger and thumb of the right hand, allowing them to slide on the lower part while grasping the upper part firmly. This will allow the model to be accurately applied to any object that is to be measured. Use the apparatus to make twelve measurements of the *thickness* of the wooden block at different positions around its edge. Write down each measurement as soon as it is obtained, and do not be disconcerted by the fact that many of them may be identical. Find the average thickness of the block, carrying the result out to one more decimal place than the individual measurements, and keep it for later reference.

If a vernier caliper is arranged to give tenths of a scale-division it is possible to make a reading of half a tenth, or a twentieth, in case two adjacent lines on the vernier show equal and opposite deviations from coincidence with two lines of the scale. In this way an

ordinary barometer scale of twentieths of an inch, provided with a 25-division vernier that gives 500ths, can be read accurately to single thousandths of an inch.

156. Slide-Rule Ratios.—It has already been seen that setting a slide rule for the ratio π and picking out a point of exact coincidence will usually give a numerical value for π that has more than three-figure accuracy, *i. e.*, greater accuracy than can be obtained by the ordinary process of "estimation" with the slide rule. Numbers like the 26 and 66 on the back of the slide rule for the ratio of inches and centimetres are so chosen as to give the correct value to at least as many significant figures as can be read with the apparatus used. Some slide rules give the same equivalent as 50 in. = 127 cm. This has the disadvantage of not being quite as easy to set on the A and B scales as on C and D, but for use with a very finely graduated instrument or with one on which the scales are 20 inches long instead of 10 it has the advantage of greater accuracy, for $66.0000 \div 26.0000 = 25.3846$ while $127.0000 \div 50.00000 = 25.40000$. The former ratio is correct to three significant figures only, while the latter is correct to five (§ 53).

157. Questions and Exercises.—1. Consider carefully what is meant by the expression "the principle of coincidence," and then write a definition of it in your own words. See that it is a definition of a *principle*, not of a process or of a condition.

2. What is the significance of the fact that the second point of "coincidence" in Fig. 48 is not as precise as the first, nor the third as good as the second?

3. Suppose the time of vibration to be just 7/8 as long for one pendulum as for another. How could you verify the fact by the method of coincidences.

4. If you take just seventeen steps while a man who

is walking beside you takes just eighteen how long are his steps in terms of yours, which are considered of unit length? How does the frequency of his steps compare with that of yours?

5. Explain why the method of coincidence is more exact than the method of estimation.

6. If the numbers in the column headed "*interval,*" § 153, are trustworthy to five significant figures how can you account for the fact that even the fourth significant figure fluctuates in column of centimeters per inch?

7. If an ordinary vernier can be read to half-tenths why can it not be used for other fractions of a tenth?

8. Could the upper scale in Fig. 48 be used as a vernier for the centimetre scale? If so, would it indicate elevenths of a centimetre, or thirteenths, or neither? How would its numbers be arranged?

XV. MEASUREMENTS AND ERRORS

Apparatus.—A vernier caliper; 100 seeds of *Phaseolus* (or other variates).

158. Direct and Indirect Measurements.—An *indirect measurement* is one that is obtained from another measurement by means of a calculation. The ordinary processes of measuring length or weight are called *direct,* because the unknown length is placed beside a standard series of multiples and fractions of the unit of length and is directly compared with it, and an unknown weight is directly balanced against a series of known weights until its exact equivalent is determined. An example of an indirect measurement is the usual method of determining density. The volume of an object and its mass are determined directly, or the volume is calculated from other measurements, and its density is then obtained by making a calculation of the ratio of mass to volume.

Are measurements of area usually direct or indirect? Can they be made in the other way?

How could the density of a liquid be determined by a direct measurement?

How could the density of a solid be determined by a direct measurement?

159. Independent, Dependent, and Conditioned Measurements.—Measurements are also classified as *independent, dependent,* and *conditioned,* and may belong to any one of these classes whether they are direct or indirect.

Two or more measurements are said to be *conditioned* if there is a theoretical relationship between them, which must always hold good. The three angles of a

plane triangle furnish an illustration of this class. Their sum must always be equal to π, or 180°.

Measurements are said to be *dependent* if one of them is allowed to influence or bias the observer when making a later measurement of the same quantity. With accurate determinations this effect is so hard to avoid, even if the observer has the best of intentions, that it is always advisable to guard against it by some such device as hiding the scale of an apparatus from view until after the indicating mark has been set in the position that has to be read. Measurements are also dependent if any essential step in making them is not repeated in successive determinations but is assumed to have its effect remain unchanged during the series. Thus, in making independent measurements by the method of coincidences (§ 153) the scales of length were not held in one position while several coincidences were found, but were reset after each determination.

It is customary to use the term *independent* only for measurements that are at the same time neither dependent nor conditioned.

If the density of an object should be determined by a direct measurement would its mass, density, and volume be independent, dependent, or conditioned?

160. Harmony and Disagreement of Repeated Measurements.—If the same object is measured several times in succession the measurements will in general differ from one another, but the differences will tend to be small under either of two different circumstances: (*a*) if the quantity is of a sharply defined character, or (*b*) if the method of measurement is coarse or crude. Thus, if a length of woolen cloth were compared with an accurate millimetre scale it would probably be found difficult to measure it the same twice in succession; but if the

same length of steel rail were clamped on rigid supports and measured at a constant temperature it might be hard to obtain two measurements that would differ. The length of one object would be a poorly defined quantity; that of the other, a sharply defined quantity.

Although sharpness of definition of the quantity to be investigated means that repeated measurements will tend to harmonize closely, it should be carefully noted that accuracy of the methods or means of measurement has just the opposite result; it is the rough methods of measurement that make repeated determinations identical with one another, and the refined methods that show discrepancies. Two similar 1-lb. weights may appear to have precisely the same mass on a rough balance and yet differ when weighed on a more carefully constructed one. If the heavier weight should then be filed down just enough to make it equal to the lighter one a test with a delicate chemical balance might show not only that the two were still unequal but even that one of them alone would not weigh the same amount twice in succession.

The general statement can be made, then, that an accurately defined quantity or a coarse method of measurement will result in a series of determinations being harmonious or identical, while a poorly defined quantity or an accurate method of measurement will cause the results to disagree or diverge.

Which measurements do you think would be most apt to show variations among themselves, the measurements of the wooden block which you made with the cardboard caliper, or measurements of the same block made with a steel vernier caliper?

It is a general truth that, no matter how sharply defined a quantity may be, the use of the most precise

methods will result in successive equally careful measurements of it differing perceptibly from one another, although the differences may be very small. Thus, the most accurate determinations of the length of a national prototype meter (§ 19) would be apt to differ by several tenths of a micron.

If equally careful measurements of the same quantity fail to agree and there is no reason for preferring one rather than another it necessarily follows that *the true magnitude of a measured quantity is always unknown.* Accordingly the various approximate measures are summarized, for actual use, by stating their *average* (the sum of n numbers, divided by n) or some other representative value, instead of by choosing one of them at random.

161. Errors of Measurement.—The *error* of a measurement is the amount by which it differs from the true value of the quantity which is measured. If the true value is always unknown the error must likewise be unknown. Such errors, however, can be discussed theoretically, and in this way much can be learned about the best manner of dealing with them.

Errors are usually classified as *constant* and *accidental*, but what are known as *mistakes* really belong in a separate class by themselves. *Constant errors* affect all the measurements of a series in the same manner or in the same direction. *Accidental errors* are *small* errors that make one measurement a trifle too large and another too small, but do not tend to bias the average result. *Mistakes* are occasional errors that are due to a lack of mental alertness on the part of the observer.

CLASSIFICATION OF ERRORS.

CONSTANT:

THEORETICAL: usually calculable, such as the faulty length of a linear scale, due to its expansion from heat.

INSTRUMENTAL: due to faulty graduation or adjustment of an instrument.

PERSONAL: some persons have a constant tendency to estimate the instant of an occurrence a little too early; others a little too late.

ACCIDENTAL:

INSTRUMENTAL: due to varying external influences, "play" of moving parts, inconstant sensitiveness, etc.

PHYSIOLOGICAL: the senses of sight, touch, etc., have a limit of sensitiveness, and this limit is not always the same.

PSYCHOLOGICAL: very likely the deductions about the outside world that result from the effect of sense-impressions on the mind do not correspond to the latter so closely as to be absolutely free from irregular variations when dealing with minute quantities.

MISTAKES:

MANIPULATIVE: doing the wrong thing.

OBSERVATIONAL: observing the wrong thing.

NUMERICAL: recording the wrong numbers. (It is especially important to guard against focussing the attention so closely on the minute part of a measurement, *e. g.*, the estimation of tenths of the smallest scale-division, that a mistake is made in recording the figures that express the larger part of it.)

162. Accidental and Constant Errors.—The difference between constant errors and accidental errors can be easily understood from the analogy of shots fired at a target. In the diagram the average position or center of clustering has a *constant* tendency townward and to the right of the bull's-eye, while the individual shots have *accidental* tendencies which carry them to one side of this average position as much as to the other. Any single shot can be considered to have a total error which is the "vector sum" or geometrical combination of its own accidental error plus the constant error of the whole group. Physical measurements are target shots in which the center of clustering can be found but in which the position of the bull's-eye is unknown.

Constant errors can be avoided or reduced to a minimum by the help of theoretical knowledge (effect of gravity in deflecting the shots downward), and by changing observers, methods, and conditions (repeating the shots when the wind blows to the left instead of to the right), and above all by judgment, experience, care, and alert hunting for all possible sources of error.

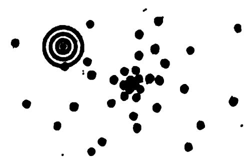


Fig. 50. Illustration of Errors of Measurement. — The shots (measurements) aimed at the bull's-eye (true value) show a general drift (constant error) to the right and downward, and individual deviations (accidental errors) that tend to extend equally in all directions from the center about which they cluster.

Accidental errors are much more easily investigated, the chief problems from the standpoint of physical science being the determination of the center of the cluster and the measurement of the degree of scattering which takes place around it.

State one *source of error* that was present when you performed each of the experiments mentioned in the following list. For example, in (*e*), one source of error would be the fact that the periphery of the disc is not an exact circle. This would give rise to a constant error in the determination of π if the diameter should be measured each time along one marked radius, but would cause an accidental error if many different diameters should be measured. Mark them in such a way as to show which ones gave rise to constant errors, and which ones caused accidental errors. Were there any mistakes?

(*a*) The measurement of your span.

(*b*) The measurement of an irregular area, first method.

(*c*) The measurement of the density of an irregular solid.

(*d*) The measurement of the sine of an angle that has a given tangent.

(*e*) The experimental determination of π.
(*f*) The calculation of e^{-x^2} for different values of x.
(*g*) The use of the formula $1/(1 + \delta) = 1 - \delta$.
(*h*) The "black-thread" determination.

163. Errors and Variations.—The theory of *accidental errors* runs closely parallel with the theory of variation in natural objects, so that a statistical investigation of the properties of several objects of the same kind, or *variates* as they are technically called, will be found to illustrate many of the facts of accidental variation which could otherwise be observed only by the more tedious study of the deviations of repeated measurements of the same object.

164. Measurement of Variates.—Examine the vernier caliper, and if it has an extra scale that reads backwards, or one scale for internal measurements and another for external ones decide which scale reads the internal distance between the jaws of the caliper, and which point marks its zero when the jaws are closed. See that you understand the vernier and have no trouble in reading it at any setting.

cm.	*frequency*
1.56	/
1.58	/
1.59	
1.60	/
1.61	//
1.62	/////
1.63	///// ///// /
1.64	///// ///// ////
1.65	///// //
1.66	///
1.67	///// /
1.68	//

EXAMPLE OF A FREQUENCY DISTRIBUTION.—Notice that the expected lengths are tabulated in numerical order before the actual measurements are made.

Measure the length of 100 seeds or other variates of the same kind. Make a table of preliminary measurements of ten variates in the order in which they were measured to find their general range, and then make a table of lengths *arranged in numerical order* likethe above, recording each different

length and the number of times that it occurs. Do not merely write the length when one seed is measured and add a pen-stroke when another of the same length is found, but see that *each* variate is represented by a mark in the second column of your table. If an unexpectedly large or small value is found after the first column has been written down it may be noted anywhere at the beginning or end of the table, as shown here.

Plot a graphic diagram in which the first ten variates have their measurements in centimetres laid off along the x-axis and the frequency of each length is represented by the height of a corresponding ordinate. Do not begin the scale of x-values at zero, but allow one square for each hundredth of a centimetre. The y-scale in any statistical diagram should measure one square for each unit of frequency, since there are rarely enough measurements to require a condensed scale and nothing would be gained by an expanded scale where there are no fractional values to be plotted. Connect the points of the

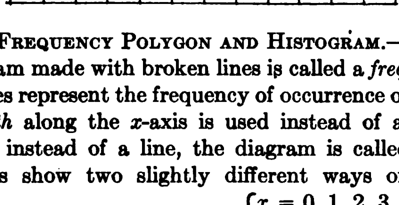


FIG. 51. FREQUENCY POLYGON AND HISTOGRAM.—The ordinary graphic diagram made with broken lines is called a *frequency polygon* if the ordinates represent the frequency of occurrence of the abscissæ. When a *length* along the x-axis is used instead of a *point*, and a vertical strip instead of a line, the diagram is called a *histogram*. The diagrams show two slightly different ways of representing precisely the same facts, namely, $\begin{cases} x = 0, 1, 2, 3, 4, 5. \\ y = 0, 2, 1, 3, 2, 0. \end{cases}$ Notice particularly the difference in the placing of the scale numbers.

diagram by a broken line and notice that the "curve" is highest in the middle and slopes downward, more or less uniformly, toward the ends. A graph of this sort is called a *frequency polygon.*

Make another graphic diagram from the same table, but instead of representing successive ordinates by the successive vertical ruled lines of the cross-section paper let them be represented by the successive white strips that lie between the vertical lines (Fig. 51), and use a vertical scale of one square for each measurement as before. A graph constructed in this way is called a *histogram.* Notice that *the area inclosed is numerically equal to the entire number of measurements or other statistics that are represented.*

Make a histogram from your table of the measurements of 100 variates.

165. Questions and Exercises.—1. In addition to the sources of error that you have stated for the experiments listed at the end of § 162, write down at least ten more sources of error that were present when you performed those experiments.

2. Which form of frequency diagram do you consider to be the more "graphic," the histogram or the frequency polygon?

XVI. STATISTICAL METHODS

Apparatus.—Ruler; slide rule.

166. Frequency Distributions.—A tabulation of a set of measurements that shows how many times each observed value occurs is said to give the *frequency distribution* of the measurements, and a graphic diagram in which the ordinates give the frequencies of the measurements that are represented by the abscissæ is called a *frequency polygon* if drawn with the usual broken line or a *histogram* if constructed by building up rectangles on successive segments of the base line.

Examine Fig. 52 and test it with a ruler held hori-

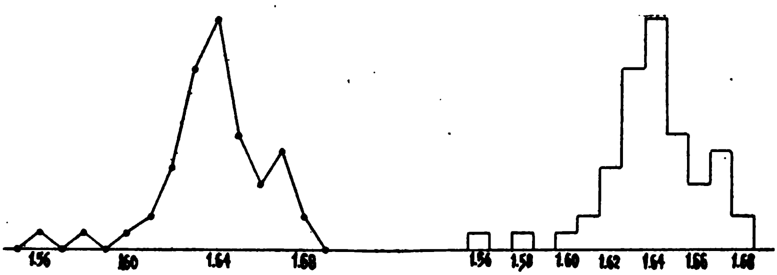


FIG. 52. EXAMPLE OF FREQUENCY DIAGRAMS.—These correspond to the table in § 164.

zontally in order to determine whether both frequency diagrams represent the same frequency distribution. Then compare, without measuring, the apparent heights of the successive points on the frequency polygon with the successive values given in the table in § 164. Has any mistake been made in attempting to represent that table graphically by the construction of Fig. 52?

167. Class Interval.—When a set of variates was measured according to § 164 the determinations were rounded off to the nearest hundredth of a centimetre by the inability of the apparatus to measure them more accurately. If the five seeds, for example, which measure 1.62 cm. in the printed table could have been measured with a much higher degree of accuracy there can be no question that each of them would have a different length from any of the others.

Under such circumstances the ordinary type of frequency diagram could never be obtained, for the frequencies would invariably be something like the series
0, 1, 0, 0, 1, 0, 1, 0, 1, 1, 1, 0, 1, 0, 1, 1, 0, 1, 1, 0, 1, 0, 1, 0, and the diagram would no longer have its characteristic "mound-like" shape with the ordinates high in the middle of the graph and dwindling away toward each end.

The shape of a frequency diagram, accordingly, may depend upon the numerical distance between the successive abscissæ that are used, the least difference between recorded measurements. In the table of § 164 this amounts to one hundredth of a centimetre, but if those lengths had been rounded off to the nearest half millimetre the frequency distribution would have become

1.55	1.60	1.65	1.70	*centimetres,*
1	9	41	2	*cases,*

in which little is left of the original shape of the graph except a suggestion of the general "mound-like" character.

The least difference in successive recorded measurements is .05 cm. in the last illustration, .01 cm. in the original table and Fig. 52, and is called the *class interval.*

If your frequency polygon tends to the nondescript type seen when the class interval is too small the meas-

urements of the hundred variates should be regrouped. Make a new *table* in which the values from 1.55 to 1.65 are all called 1.6, those from 1.65 to 1.75 are considered as 1.7, etc. If several measurements have been recorded as 1.65 put about half of them in the class 1.6 and the other half in the class 1.7. From this table plot separately both the frequency-polygon and the histogram that illustrate it graphically.* Do not use an expanded scale in any statistical graph either for frequency or for class interval.

168. Types of Frequency Distribution.—If the class interval is made very small and the measurements are very numerous, both forms of diagram can be considered as losing their abrupt changes until they merge into two identical curved lines, the notches that were originally present having become indefinitely small. Frequency distributions are not invariably "mound-shaped," but may be classified, according to the general shape of the graphic diagram, as (*a*) *symmetrical,* (*b*) moderately *asymmetrical,* (*c*) very asymmetrical or *J-shaped,* (*d*) bilocular or *U-shaped,* (*e*) *rectangular* (Fig. 53).

The symmetrical type (*a*) is seen in physical measurements. Notice that in it (1) the average (*i. e.,* middle) x-value is the most frequent; (2) measurements that are a given amount above or below the average are less frequent than it but of equal frequency with each other; and (3) extremely divergent values do not occur. The asymmetrical type (*b*) is similar except that the most frequent value does not lie in the middle of the distribution and on one side of it the frequency falls away more rapidly than on the other. Moderate asymmetry is common in all statistical data. An example of

* If preferred, a class interval of .05 cm. may be used instead. This will avoid splitting any original class into two.

the J-shaped type (*c*) may be seen in the distribution of wealth among any population; the frequency of individuals with little wealth is very great, and with in-

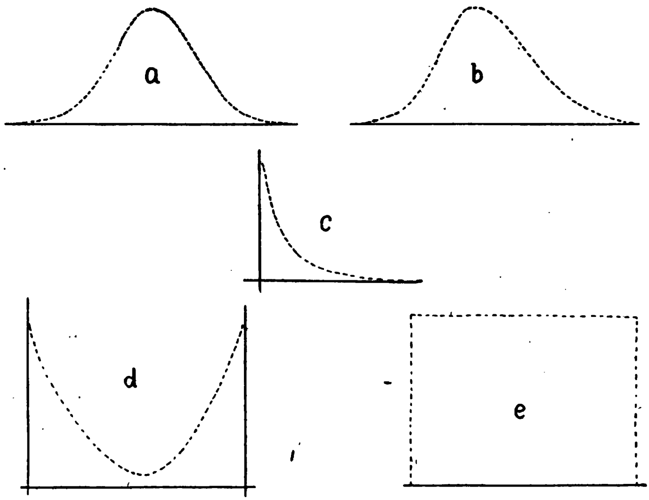


FIG. 53. TYPES OF FREQUENCY DISTRIBUTION.—(*a*) Symmetrical. (*b*) Positively asymmetrical (skewed toward the right). (*c*) Extreme asymmetrical, or J-shaped. (*d*) Bilocular, or U-shaped. (*e*) Rectangular.

creasing wealth the number of cases falls off until it reaches a vanishing point. The curious U-shaped type (*d*) is seen where there are tendencies toward both extremes, or the centre is a position of unstable equilibrium; and the rectangular type (*e*) occurs in purely mathematical cases, such as the actual error of a tabulated logarithm, which is never more than $\pm$ 0.5 of the unit in the last decimal place and has all intermediate values with equal frequency.

169. The Probability Curve.—The measurement of 100 seeds will probably show a moderate degree of asym-

metry, since such objects do not grow beyond a definite size but do fall short of it in many cases. This is called *negative asymmetry*, and the curve is said to be skewed toward the left. *Physical measurements, however, tend to be above the average just as often as they are below it*, and so give the symmetrical form of frequency distribution. As the measurements are made more and more numerous the frequency polygon approaches more and more closely to the form of the so-called *probability curve*, $y = e^{-x^2}$, which was calculated in the lesson on logarithms and plotted in the lesson on graphic representation. In some cases it will appear drawn out relatively flat and in others will be very high and narrow (Fig. 34), but the curve is the same in all cases, except that the scales of x-values and y-values are condensed or spread out to different degrees.

Carry out the binomial expansion that is given below at least as far as $n = 15$, noticing that each two successive terms have to be added in order to obtain the term below and between them. Lay off a series of equal intervals on the x-axis. On these erect a series of ordinates proportional to the successive terms of the last polynomial, *in order*. A smooth curve through the tops of the ordinates will give a very good approximation to the probability curve.

$$
\begin{aligned}
(1+1)^0 &= 1 \\
(1+1)^1 &= 1+1 \\
(1+1)^2 &= 1+2+1 \\
(1+1)^3 &= 1+3+3+1 \\
(1+1)^4 &= 1+4+6+4+1 \\
(1+1)^5 &= 1+5+10+10+5+1 \\
(1+1)^n &=
\end{aligned}
$$

170. Representative Magnitudes.—For most scientific work, the statement of a whole frequency distribution

or of each one of a long series of measurements would be a cumbrous process and reading such statements would be a tedious task. Accordingly it is customary to summarize such a set of values by stating some representative value, such as the average. In special cases such representative magnitudes as the *geometrical mean*, $\sqrt[n]{(a_1a_2\cdots a_n)}$, or the *harmonic mean*, $n/(1/a_1 + 1/a_2 + \cdots + 1/a_n)$, or the *quadratic mean*, $\sqrt{[(a_1^2 + a_2^2 + \cdots + a_n^2)/n]}$, have been employed, but the most usual one is the *arithmetical mean* or *average*, $(a_1 + a_2 + \cdots + a_n)/n$. The *median*, which is a_n if $a_1, a_2 \cdots a_{2n-1}$ are arranged in order of size, is frequently useful as a representative value, as is also the *mode* or *modal value*, which is simply the value that occurs with the greatest frequency. The word *mean* is sometimes used in a general sense for *any representative value*, and sometimes is restricted to the same significance as the word *average*.

171. The Average.—The average is obtained by adding together a set of values and dividing the sum by the number of values. Thus the average of the five values 3, 3, 4, 5, 10, is one fifth of their sum, or 5. If certain values occur repeatedly the average is obtained by dividing the sum by the number of values, not by the number of different values. Thus, the average of the measurements given in the table at the end of the preceding lesson is $(2 \times 1.68 + 6 \times 1.67 + 3 \times 1.66 \cdots)/(2 + 6 + 3 \cdots)$, or 8686/53 or 1.639.

172. The Median.—The median is obtained by choosing such a value that half of the other values exceed it and half are below it. *If the numbers are arranged in numerical order* in a column the number that is half way down the column is the median. Otherwise it may be found by crossing off the largest number and the smallest,

and repeating the process until only one is left. If two numbers are left, as will be the case if there are an even number of measurements, the number half way between them can be taken as the median, but in physical or statistical data it usually happens that the two remaining numbers are the same.

Find the median of 3, 3, 4, 5, 10. What is the median of 7.4, 6.8, 7.3, 7.3, 7.2, 7.1, 7.2? Ans.: 7.2. Find the median of the measurements that are tabulated in § 164.

Turn to the histogram in Fig. 51, § 164, and cross off one of the small squares between the curve and the base line at the extreme left and then one at the extreme right; repeat the process until only one (or two) squares are left, and show that the median is 3. Imagine that Fig. 54 is similarly divided up into small squares, each square representing one measurement, and satisfy yourself of the truth of the fact that *the ordinate that is drawn upward from the median x-value must bisect the area of the frequency curve.*

173. The Mode.—The mode is the most frequent value. Thus in the set of values 3, 3, 4, 5, 10 the mode is the number that occurs twice, namely 3. In the illustration of the measurement of variates the mode is 1.64, the length that was found most often.

174. Choice of Means.—If a frequency distribution is of the *symmetrical* type the average, mode, and median will all be the same.

Where there is a *moderate degree of asymmetry* the median will come nearer to the mode than the average does, as is shown in the following diagram. The mode is easily seen to be the most probable value. If a seed is taken at random from the set whose measurements were given in the table it is more likely to measure 1.64 cm. than any other amount. The mode has one

decided disadvantage, however, in that it cannot always be chosen from a given frequency distribution. For example the mode cannot be obtained from the series

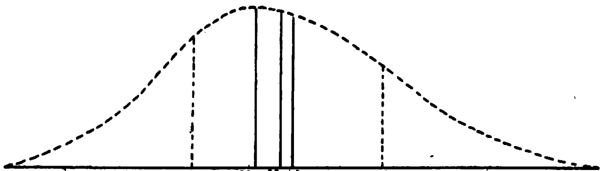


FIG. 54. TYPICAL FREQUENCY DIAGRAM.—The mode has the highest ordinate. The ordinate of the median bisects the area under the curve. The ordinate of the average passes through the center of gravity of the area. The median lies between the mode and the average, and is twice as far from the former as from the latter. The difference (distance) between the quartiles Q_1 and Q_3 (§ 178) is called the *interquartile range.*

3, 4, 4, 5, 6, 6, 7, 8, except by assuming that the values follow the probability law, and fitting a probability curve to them as accurately as possible by a "black thread" method; the position of the top of this theoretical curve can then be determined. Where there is a long series of measurements the average is sometimes calculated from the formula, indicated on the above diagram, that $\overline{me} - mo = 2\ (\overline{av} - \overline{me})$, or that *the median lies $\frac{1}{3}$ of the way from the average to the mode.*

The median, like the mode, is easy to determine. It can be used in two classes of cases where the average cannot be determined. One is in case measurements have been tabulated with indefinite terminal classes, *e. g.*, "less than 10 mm., 7 cases; 10 to 12 mm., 4 cases; 12 to 14, 5; 14 to 16, 3; above 16, 2 cases." Here the median is the class "between 10 and 12," say 11 mm., and the mode is probably the class "12 to 14," say 13 mm., but the average cannot be found on account

of nine of the numerical values not being stated; the best that could be done would be to guess that the average was a little less than the median. The other case is where quantities can be arranged in numerical order but are difficult to measure individually. Thus it may be difficult or impossible to gauge the scholarship of a student in accurate numerical terms, but if a group of students can be arranged in order of scholarship the median can be determined without difficulty. Furthermore the median has a unique advantage over most other representative magnitudes in that it is not affected by inverting the unit of measurement. The median of a number of prices will be the same whether they are given as cents per dozen or as dozens per dollar. Of a group of different velocities the same one will be picked out by choosing the median whether the numerical values are expressed in miles per hour or in minutes for a one-mile run. It is easy to see that this will not be the case if the average is used. The median, however, is not quite as good a representative value as the average, in case they are different, for a series of measurements that have all been made with equal care. Curiously enough, however, the more extensive a series of measurements the more likely it is to show that the individual measurements are not equally trustworthy but may be grouped in different classes according to their relative scattering. It is in such cases that the median is a much better representative figure than the average, for the average is influenced by an unduly large or small measurement just in proportion to the aberration of the latter, while as long as a measurement is above the median it makes no difference how far above it may be, its effect is no greater than that of any other single value. (Compare the average and the median of 3, 3,

4, 5, 10, with those of 3, 3, 4, 5, 30. The very erratic values ought to have the least influence instead of the most.)

175. Deviations.—Of almost as much importance as finding the best representative value for a series of measurements is the determination of how closely they cluster around it or how widely they scatter from it. This will help to furnish information in regard to the accuracy of the measurements, and hence also in regard to the accuracy of the instruments and methods employed in making them; it will also be useful in comparing and combining determinations made at different times or by different observers. The difference, $m - a$, between any single measurement and the average (or other representative magnitude) is known as the *deviation* or *variation* of that measurement and is denoted by the letter v. It is not the same as the error of the measurement, for the error may have a constant component that affects all of the measurements equally; but it may be considered as the accidental error or accidental component of the total error. Deviations give no positive indication of any constant errors that may be present.

If each one of a series of independent measurements is as trustworthy as any other it can be shown mathematically that their best representative value is their arithmetical mean, or average; and accordingly the average is the figure that is almost invariably used.

Copy the two following columns of figures, find the average of each set, and write after each measurement (m) its deviation (v) from the average of the figures in the same column, marking it with a minus sign if the value is less than the average, and with a plus sign if in excess of the average.

Although both averages are the same it is evident that the first set of measurements must have been made by a more trustworthy instrument, observer, or method than the second. If the averages had not been of the same value the first one would undoubtedly have been entitled to more confidence, other things being equal, than the second.

m	*v*	*m*	*v*
27.35		27.36	
27.34		27.38	
27.34		27.35	
27.33		27.37	
27.34		27.32	
27.34		27.30	
27.33		27.31	
27.34		27.30	
27.35		27.35	
27.34		27.36	

Add each column of deviations and verify the fact that *the algebraical sum of the deviations from the average is always zero.* If their sum is zero their average will also be zero, so that when an "average deviation" is recorded the term means not the average of the algebraical deviations but the average of the values that they would have if the negative signs were omitted.

176. Average by Symmetry.—The foregoing property suggests an easy method of finding the average in simple cases: If a number can be so chosen that the individual measurements are symmetrically grouped around it the sum of the positive deviations will equal the sum of the negative deviations and the number will be the required average.

What is the average of 115 and 119? Ans.: 117, because it makes the sum of the deviations (+ 2 and − 2) equal to zero.

Find the average of 3, 6, 9, 12, 15 by the method of symmetry.

What is the average of 16, 18, 20, 22? Of 14 and 17? Of 12, 14, 17, 19? Of 126.8 and 127.4? Of 121 and 141? Of 161 and 191? Of 198, 199, 203? Of 8, 10½, 11½? What is the value, to the nearest whole number,

of the average of 117, 116, 117? (Suggestion: Is the average above or below 116.5?) What is the exact average of 17, 18, 18, 19, 19, 20?

177. Average by Partition.—Before leaving the subject of the average it should be noted that there is no need of adding the entire numerical values if they contain a part in common. Thus in either of the columns of figures in § 175 the average is obtained by writing the first three figures, which are common to all values, and annexing the average of the last figure.

178. Quartiles.—Just as the middle number of a series arranged in ascending or descending order of magnitude is called the median, or half-way value, so the middle one of the numbers that lie below the median is called the *lower quartile*, or quarter-way value; and the median of the numbers that lie above the median is known as the *upper quartile*, or three-quarter-way value. The quartile abscissæ are laid off on the base line of Fig. 54, § 174, at Q_1 and Q_3. It should be carefully noticed that their ordinates, together with the median ordinate, divide the whole area of the frequency diagram into four equal parts. If there is any difficulty in understanding this remember that the area of a histogram indicates the total number of measurements (§ 164). If half of them lie below the median, by definition, and half above there can be no trouble in realizing that the median is the abscissa whose ordinate bisects the area of the figure. Of course the same thing is not true of the average, except when average and median happen to have the same value.

What are the quartiles of the series 3, 4, 4, 5, 6, 8, 11?

Find the median and quartile values of the set of numbers 28, 29, 31, 36, 31, 30, 35, 33, 32, 36, 29, 28, 32. Notice that the numbers "below" the median need not

all be smaller numbers; the median is 31 and one of the lower numbers is also 31. Notice that the lower quartile is the median of the six numbers *below* the median, not of the *seven* numbers that include the median itself. Similarly, the upper quartile is not 33, but 34 (§ 172).

179. Semi-Interquartile Range.—The numerical distance between the lower quartile and the upper quartile is called the *interquartile range.* In the series 7, 8, 10, 13, 19, 22, 27, the quartiles are 8, and 22, and their difference, 14, is the range between quartiles, or interquartile range. Half of this, or 7, is called the *semi-interquartile range;* notice that it is the average of the two distances from median to quartile. It is sometimes used as a measure of the scattering or clustering of a set of observations, and its theoretical importance will be seen later.

Find the semi-interquartile range of each of the two columns of figures in § 175 after re-arranging the data in numerical order. For the first column: if the five smaller values are considered as lying below the median and five above, that is, if the median is considered to occupy no numbers in the middle of the column the lower quartile will be 27.34; but if only four values are considered as lying below the median, and four above, that is, if the median is considered to include two numbers in the middle of the column the lower quartile will be 27.335. As the median is actually considered to be neither two numbers nor no numbers, but is one single number it will be satisfactory to say that the lower quartile is neither 27.340 nor 27.335, but is half way between these two values, say 27.338. Similarly, show that the upper quartile is 27.342. The second column should be treated in the same manner. What do you notice about the relative size of the two semi-interquartile ranges?

180. Questions and Exercises.—1. Complete the following rule for finding the upper quartile of 12 measurements, and submit it to your instructor for inspection: "Arrange the measurements in order of magnitude and find the median by taking the average of the sixth and seventh. Then . . ."

2. When the tabulated values of § 164 are re-grouped in larger class intervals (§ 167) there are seven measurements of 1.65 cm., which theoretically should be separated into two equal groups for inclusion with the classes 1.60 cm. and 1.70 cm. Would it be objectionable to add exactly $3\frac{1}{2}$ squares to each of the strips of the histogram, or would it be better to add 3 to one and 4 to the other? Explain why.

3. Why does it generally happen that the median of a set of measurements fails to be exactly twice as far from the mode as from the average?

4. Draw roughly a curve like Fig. 54, but having negative instead of positive asymmetry. Place the mode and the median where you think they should go, and then locate the average.

5. Make a tabular statement of (*a*) the advantages of the median over the average, and (*b*) the advantages of the average over the median, as a representative value.

6. *Prove* that for any three numbers (say, $k + d_1$, etc.) the algebraical sum of the deviations from the average must be equal to zero.

7. If a train travels the first, third, fifth, seventh, . . . miles at a rate of 20 miles per hour, and the second, fourth, sixth, eighth, . . . miles at 30 miles per hour, show that its average velocity is not the average of the two rates, 25 miles per hour, but is their harmonic mean (§ 170) instead.

8. What would you suggest for a *numerical measure of asymmetry* in a diagram like Fig. 54?

XVII. DEVIATION AND DISPERSION

Apparatus.—Vernier caliper; variates; slide rule; table of logarithms.

181. Characteristic Deviations.—Just as the use of an average or other representative magnitude obviates the necessity of stating the separate measurements from which it is derived, so a statement of all the deviations from the average can be replaced by a single *characteristic deviation.* A set of measurements can then be summarized by two numbers, and it is customary to write such a result in the form $a \pm d$,* where the first number gives the average value of the quantity measured and the second expresses the limiting distances above and below the average which mark off in some way an amount of deviation that is characteristic for the set of measurements.

182. Total Range.—The simplest way of summarizing the deviations of a series of measurements is by stating their *total range,* or the algebraical difference between the smallest one and the largest one. Obviously the extreme range of the measurements themselves will also give the same result. Thus, in the two columns of measurements in § 175, show that the total range is .02 for the first and .08 for the second. Unfortunately, this is not only the easiest but also the worst method of obtaining a characteristic deviation, for the numerical value given by the total range depends upon only two of the measurements of the series; and those two are

* The letter d will be used in this book to denote a particular kind of *characteristic deviation.* A single deviation of any individual measurement will be indicated by the letter v (*variation*).

the least satisfactory ones, because a repetition of the series of measurements would be likely to show a considerable fluctuation in the largest and smallest values, while the most closely clustered values would hardly be changed at all.

What is the numerical value of the total range of the data in the table of § 164? In what kind of units should it be expressed?

183. Average Deviation.—A better index of the amount of scattering is given by the *average deviation.* This is the average of the positive arithmetical values of all of the deviations. The true algebraical average of the deviations cannot be used if they have been calculated from the average measurement because, as has been shown (§ 175), its value is always zero.

The average of the positive values of the deviations is always smallest when the deviations have been calculated from the median measurement, and it is in connection with the median that the average deviation is generally used.

Find the average and the median of the numbers 3, 3, 4, 5, 10. Then find the deviation of each of these numbers *from the median,* being careful not to omit one of the deviations in case it happens to be zero. What is the value of the average deviation when calculated from the median? Find also the five deviations *from the average,* and determine the average of their (positive) arithmetical values. Is the average deviation also smaller when measured from the median than when measured from the mode?

184. Standard Deviation.—The typical deviation that is most frequently used for statistical purposes is the *standard deviation.* This is the square root of the quotient of the sum of the squares of the deviations from

the average divided by one less than the number of statistical values.* If there are n quantities whose deviations from their average are respectively $v_1, v_2, v_3, \cdots v_n$, the standard deviation of those quantities is the value of

$$\sqrt{\frac{v_1^2 + v_2^2 + v_3^2 + \cdots + v_n^2}{n-1}}.$$

The standard deviation is a special case of the *mean-square deviation* (or *root-mean-square deviation*), being in fact the mean-square deviation from the average, or approximately the quadratic mean of the deviations from the average. It is also sometimes called the *mean deviation* and the *mean-square* deviation, but as the term *mean deviation* is also used for what we have defined as the *average deviation* it is much better to avoid the use of the word *mean* (§ 170) altogether in connection with a deviation.

185. Dispersion.—The measure of deviation which is used for physical measurements in almost all cases is that particular form of characteristic deviation which is called the *dispersion*. It is approximately two thirds of the standard deviation, or, more exactly,

$$\pm .674490 \sqrt{\frac{v_1^2 + v_2^2 + \cdots + v_n^2}{n-1}}.$$

186. Significance of the Dispersion.—An important characteristic of this typical value is that for a series of measurements which is extensive enough to follow the

* To be strictly accurate, the standard deviation of the statistician is $\sqrt{\Sigma v^2}/\sqrt{n}$. The reasons for using $n - 1$ will be found in the larger treatises; here it need only be noticed that the denominator is diminished because the numerator is smaller than the sum of the squares of the true errors (§ 216). Of course, if n is fairly large, the distinction is unnecessary.

law of the probability curve *the dispersion can be shown mathematically to express exactly the same limits above and below the average as are given with respect to the median by the semi-interquartile range.** If measurements follow the law of the probability curve the median and the average will coincide; and the upper and lower quartiles will include within the space between them just half of the total number of measurements, as was seen in connection with Fig. 54. In other words, the right-hand half of the frequency diagram will be bisected by the ordinate corresponding to the positive value of the dispersion, and the corresponding abscissa will be the median value of all the positive deviations. Since the left-hand half of the curve is likewise bisected by the negative value of the dispersion, and the graphic diagram is symmetrical with respect to the y-axis, it follows that the dispersion is the median of the absolute arithmetical values of all the deviations, or that *any single deviation is as likely to be less than the dispersion as it is to be greater.* The dispersion, with its plus-and-minus sign marks off a small distance above and below the average, and it is just as likely as not that any single measurement chosen at random will be within these limits.

187. Advantage of the Dispersion.—Where the number of measurements in a series is relatively small, say not greater than 10, the dispersion obtained by calculation and the semi-interquartile range obtained by picking out the quartile values will not usually have exactly the same value, on account of the measurements not being sufficiently numerous to follow the laws of probability

* Show that the median and semi-interquartile range of the 53 measurements given in the table of § 164 are 1.64 cm. and 0.01 cm. Could these two numbers be used as a rough approximation for *average* and *dispersion?*

very closely. Even in this case, however, the *value given by the formula* is preferable to half the difference between the quartiles because the former is obtained from all the measurements of the series and the latter from only two. Where there are as many as ten measurements of a physical magnitude it is usually found that the two values will not differ by more than 15 or 20 per cent., and for rough preliminary measurements the semi-interquartile range is almost as satisfactory as the dispersion and can be obtained much more readily.

m	v	v^2
1.82 *cm.*	33	1089
1.85	63	3969
1.85	63	3969
1.78	7	49
1.82	33	1089
1.84	53	2809
1.78	7	49
1.62	167	27889
1.81	23	529
1.70	87	7569
17.87 (10		49010 . . . $\Sigma(v^2)$
1.787 . . *av*		

$$\begin{aligned} \log 49010 &= 4.6903 \\ \log 9 &= \underline{\ .9542} \\ 2)&\underline{3.7361} \\ &\ \ 1.8680 \\ \log .6745 &= \underline{1.8290} \\ &\ \ 1.6970 = \log d \\ d &= 49.78 \end{aligned}$$

188. Calculation of the Dispersion.—The table shows the measurements (m) of ten variates taken at random from the 100 that were measured before, also their average (av), and their individual deviations (v) written without decimal points.* The third column shows the squares of the numbers in the second column, as obtained from the *table of squares* (see appendix). According to the formula in § 185 the sum of these squares is to be divided by 9, the square root of this quotient is then to be found and multiplied by about $\frac{2}{3}$ in order to obtain the dispersion of this set of measurements.

* In the calculation of a dispersion the deviations should always be written down as whole numbers. Nothing would be gained by writing these v's in centimetres instead of in thousandths of a centimetre.

If the numbers in column v are expressed in thousandths of a centimetre the dispersion will also come out in thousandths; thus, the value of d shown in the illustration represents .04978 cm. This means that the limits 1.737 cm. and 1.837 cm. are the boundaries between which about half of the measurements should lie (see § 186). Verify this from the tabular values of m.

189. Rule for Accuracy of the Average.—In determining how many figures of the average are to be retained as significant it is best to follow the rule that *at least half of the deviations should be greater than* 3. In the illustrative example notice that keeping three decimal places in the average has made more than half of the deviations greater than 30. In such a case the average could obviously be rounded off to *two* decimal places, 1.79, and at least half the deviations would still be greater than 3. This will be found to simplify the calculation and the result will not be essentially different.

Find the dispersion of the same ten variates by writing a column of deviations from the value 1.79, and calculating the value of .6745 times the square root of one ninth of the sum of their squares. Use logarithms but *do not* refer to the illustrative example in § 187 for each step of the process; refer to the formula (§ 185) instead.*

Notice that your final result agrees with the one previously obtained as far as three significant figures. This is ample accuracy, since two significant figures are all that is usually wanted for the value of a dispersion.

190. Use of the Table of Dispersions.—The root extraction and long multiplication and division should, of course, never be done by the tedious arithmetical process. Even the logarithmic process used in the illus-

* Referring to the steps of a typical example is almost always the easiest way of *obtaining a required result;* it is a very poor way of *learning a method of procedure.*

tration can be avoided as follows: $49010 \div 9 = 5446$; the square root of this will be somewhat more than 70; $\frac{2}{3}$ of 70 is about 48; in the column headed $(n \pm \frac{1}{2})^2/.6745^2$ of the table of squares in the appendix find two numbers between which 5446 lies and read the corresponding number in column n. It is immediately seen to be 50, which agrees with the previously obtained 49.78 as far as the two significant figures which are all that is required. As any arrangement of significant figures has two square roots (see § 13, no. 41, and § 86) a place for 5446 would also be found opposite 16 in the table, but the rough preliminary check-calculation showed that the answer should be about 48, so there could be no doubt that the required answer is 50 rather than 16.

If the sum of the squares had been 490100 what would be the value of the dispersion, pointed off so as to give centimetres of length? Use the table of squares in the manner just illustrated.

If the sum of the squares of the deviations of *sixteen* variates is 49010 what is their dispersion?

191. Dispersions with the Slide Rule.—In the rest of this course the dispersions are to be calculated either with the table of squares, as explained, or with the slide rule, which makes the process even easier. A special line is marked at 6745 on the C scale so that $.6745\sqrt{a/b}$ can be obtained with a single setting as soon as the sum of the squares of the deviations is obtained. Difficulty with the double square root is best avoided by setting the end of the C scale to the approximate value of the radical as obtained by a rough mental calculation; a very slight movement of the slide is then all that is needed to make an exact setting.

If the sum of the (v^2)'s of 12 measurements is 28860 find the approximate value of the dispersion mentally,

and then find the exact value with one setting of the slide rule.

192. Sigma Notation.—The capital letter sigma (Σ) of the Greek alphabet is often used by mathematicians, prefixed to an algebraical term, to denote *the sum of all such terms*; thus the expression for the average, namely, $(a_1 + a_2 + a_3 \cdots + a_n)/n$ is abbreviated to the equivalent form $(\Sigma a)/n$. In the same way the formula

$$.6745 \sqrt{\frac{v_1^2 + v_2^2 + \cdots + v_n^2}{n - 1}}$$

is more easily and compactly written in the form

$$.6745 \sqrt{\Sigma(v^2)/(n - 1)},$$

and the use of $\Sigma(v^2)$ will have been noticed previously in the example of the calculation of a dispersion.

Rewrite the formulæ for the harmonic mean and the quadratic mean (§ 170), using the sigma notation. Write the formula for $(x_1 + x_2)^2$, using the same notation as far as possible.

If five measurements of the quantity x give 3, 3, 4, 5, 10, what is the value of Σx? If the average is 5 what is the value of Σv? Of $\Sigma(\text{mod } v)$?* Of $\Sigma(v^2)$? Of $\Sigma(x^2)$?

193. Dispersion of an Average.—The average length of 10 variates has already been calculated. If 10 more were measured their average would be somewhere nearly the same as the first average. If a considerable number of such averages had been determined it would be a simple matter to determine *their* dispersion, and the result would naturally be a smaller number than the

* The *modulus* of a (real) number means its arithmetical value regardless of its sign; its "absolute" value; the positive square root of its square.

dispersion of any of the sets of individual measurements, since an average is a more trustworthy figure than a single determination. In fact it can be shown mathematically (*vide infra*) that if ten equally good measurements are averaged the single measurements will show a variation which is greater than the variation of such averages in the proportion of $\sqrt{10}$ to 1, and similarly that *the dispersion of averages will be* $1/\sqrt{n}$ *as great as the dispersion of individual measurements if the latter are averaged in groups of* n. This means that it is not necessary to calculate several averages in order to find their dispersion, for it can be determined from a knowledge of the number of measurements that go to make up a single average and from the dispersion of these individual measurements. Thus the *dispersion of the average* (d_{av}), as it is called, of nine measurements is at once seen to be one third as large as the dispersion for single measurements (d_1), since the square root of nine is three.

If a set of 16 measurements are free from constant errors how much more accurate is the average than one of the individual measurements?

The formula for the dispersion of an average is easily written, for if

$$d_1 = .6745 \sqrt{\frac{\Sigma(v^2)}{n - 1}},$$

then

$$d_{av} = .6745 \sqrt{\frac{\Sigma(v^2)}{n(n - 1)}},$$

the second formula being $1/\sqrt{n}$ as large as the first.

194. The Statement of a Measurement.—It is customary to write the result of an accurate physical measurement in the form of two numbers separated by a plus-or-minus sign. The first number is the *average;*

the second one is the *dispersion of the average,* not the dispersion for single measurements.

Tabulate the first eleven of the twelve measurements of the wooden block made with the card-board model of a vernier caliper. Pick out their quartiles and find the semi-interquartile range, to be used as a rough value of the dispersion of the individual measurements, and divide it by the square root of n. Write the thickness of the block in the form $av. \pm d_{av}$. Divide d_1 for your measurement of 10 seeds by $\sqrt{10}$ in order to obtain d_{av} for them, and state their measurement in the same form.

Make eleven more measurements of the wooden block, this time with the vernier caliper that gives tenths of a millimeter and treat them in the same way as the previous eleven. A typical set of results is shown in the margin. Two important facts are illustrated by this table: (1) It is the accurate method that shows discrepancies between repeated measurements and the rough method that shows more uniform agreement. (2) There will be nothing fallacious about the statement that a dispersion "is zero" if care is taken to *point off that zero.* The semi-interquartile range of the first column is .0 while that of the second is .01 cm., but .0 cm. is the only correct way of rounding off to one decimal place the number which is expressed in the second decimal place by the figures .01 cm.

model cal.	*steel cal.*
3.7	3.75
3.7	3.74
3.7	3.75
3.8	3.76
3.7	3.76
3.7	3.75
3.7	3.76
3.7	3.75
3.7	3.75
3.7	3.74
3.7	3.74

COMPARISON OF ROUGH MEASUREMENTS AND PRECISE MEASUREMENTS.

195. Relative Dispersion.—It has already been shown that in order to see how serious an error really is it

should be referred to or divided by the true magnitude of the quantity measured. In the same way, instead of using the actual value of the dispersion, it is often found more useful to obtain the *proportional dispersion,* or *relative dispersion,* or *fractional dispersion,* as it is also called; the ratio of dispersion to representative magnitude.

In the two measurements just stated, the length of a seed and the thickness of the wooden block, divide the dispersion of the average by the average itself in order to obtain the *relative dispersion of the average.* Express this either as a decimal or as a percentage. Notice that it is smaller for the thickness of the block, which is fairly uniform, than for the length of a seed, which varies considerably. The absolute dispersion of the average in centimetres, however, may be larger for the block than for the seeds if a large dimension of the block is measured while the seeds that are used are small.

196. Questions and Exercises.—1. If each deviation, v, is a number of thousandths explain why the mathematical operations of finding the dispersion cause d also to be given in thousandths of a unit.

2. If the results of a series of measurements are found to be 8.16 cm. $\pm$ 0.033 cm. would it be advisable either (a) to write 8.160 $\pm$ .033, or (b) to write 8.16 $\pm$ .03, instead of using the first form? Explain why.

3. Does the number 49010 in the table of § 187 mean 490.10 or 4.9010? Is it a number of centimetres or of square centimetres?

4. Re-arrange the following table so that each colloquial statement is associated with its proper numerical equivalent.

"a few hundred"	20 ± 5
"nearly a gross"	250 ± 50
"a dozen or so"	70 ± 10
"upward of three score"	130 ± 10

5. Express each of the following in the form of a representative magnitude and a measure of scattering:

"over a dozen"	"six or eight"
"nearly a hundred"	"a few"
"about a thousand"	"several"
"in the neighborhood of 15 or 20"	"some"

6. If "average deviation" means the average of the (positive) arithmetical values of the deviations, what would probably be meant by the expression "median deviation"? Has any characteristic deviation already been named which has practically the same significance?

XVIII. THE WEIGHTING OF OBSERVATIONS

Apparatus.—Platform balance; clamp and bar or stand to support the balance 40 or 50 cm. above the table; set of weights; vernier caliper; aluminum block; overflow can and catch-bucket for measuring displaced water; towel; string (80 to 100 cm.) and two spreading rods; fine silk thread; slide rule.

197. Necessity of Weights for Observations.—A representative value is often wanted for measurements which are not all equally trustworthy. The accepted values for such constants as the maximum density of water, the mechanical equivalent of heat, the length of the true ohm of mercury, the velocity of light *in vacuo*, have all been derived from measurements by different observers at various times, and in general by different apparatus and methods. Any of these varying factors will produce varying results, and one determination can sometimes be accepted with more confidence than another, and so will be entitled to greater "weight" when it is necessary to decide upon a representative value.

198. Density by Different Methods.—An example of the effect of different methods on the determination of a physical magnitude may be given by the measurement of the density of a metal block. If the mass is known this can be accomplished either by mensuration, or by measuring displacement, or by a measurement of buoyant force. According to the Principle of Archimedes the apparent loss of weight of a body immersed in a fluid is the same as the weight of an equal volume of the fluid. If the volume of a metal block is v, its weight w, and its apparent weight in water w', the density can be found

as the ratio of the weight, w, to the loss of weight. $w - w'$, supposing that the density of water is unity; or it can be determined as the ratio of the weight, w, to the measured volume of water that is actually displaced on immersion, say v'; or the block can be measured with a caliper and the density calculated as m/v.

It will first be necessary to arrange the apparatus so that the apparent weight of an object can be determined while it is immersed in water. Place the platform balance on the support or clamp it to the cross-bar above the table in such a way that an object can be weighed by suspending it under the bar with strings attached to a spreading rod that is laid on one scale-pan of the balance. See that the beam of the balance moves freely. Use the other spreading rod as a counterpoise, and make a careful allowance for the fact that they may not exactly balance.* Attach the aluminum block to the string by a fine *thread* long enough to allow it to hang within the empty overflow can, and (1) weigh it as accurately as possible. Fill the overflow can with water, closing the spout with the finger-tip; place it in position where the aluminum block is to hang, with the catch-bucket under the spout; remove the finger and allow the excess of water to run out of the overflow can; then (2) weigh the catch-bucket with its contained water, and replace it in position. Lower the aluminum block carefully into the overflow can and (3) weigh it while submerged; then (4) weigh the catch-bucket again in order to find out how much water was displaced.

Find the density of the aluminum block (a) by comparing its weight with the weight of the overflow of water actually displaced; (b) from the two values w and w'; (c) by measuring the block with the vernier caliper,

* Weigh their *difference;* there is no need of weighing each one separately.

computing its volume as closely as possible, and applying the formula for density, $d = m/v$. Report your results to the instructor for comparison with those of the other members of the class.

199. Weights for Repeated Values.—The simplest case of weighting different observations is when separate numerical values have each been obtained a definite number of times. Suppose, for example, that the density of a block of aluminum has been determined both as 2.6 and as 2.7, in the total of five measurements, the smaller value having been found on four occasions while the larger value was obtained only once. The best representative figure from these data certainly would not be the number 2.65, half way between 2.6 and 2.7, but ought to be a number situated four times as far from the least frequent measurement, 2.7, as from the most frequent one, 2.6; in other words, it should be the number 2.62. Moreover, this is easily seen to be the same result as would be obtained by taking the average of the five individual measurements. (Try it.) The rule in such a case is obviously to *give each numerical value a weight proportional to the number of times of its occurrence.*

Find the *weighted average* of the values of a measured length if it was found to be 2.345 cm. in each of six trials, 2.350 cm. in twelve trials, and 2.355 in nine trials. (Suggestion: it is a little easier to calculate the value of $2 \times 2.345 + 3 \times 2.355 + 4 \times 2.350$.)

200. The Weighted Average.—The weighted average is found in any case by considering that certain values have been obtained more frequently than others. In the case just discussed this was a fact, in other cases it is only a supposition made to fit the known or estimated intrinsic value of the observations.

If a difficult measurement had been made by an experienced student and found to be 0.35, while the same experiment gave the value 0.41 when performed by a beginner, it might be decided somewhat arbitrarily to give the first number twice the weight of the second. The process of finding the weighted average, $(2 \times 0.35 + 1 \times 0.41)/3$, would then be equivalent to supposing that the better measurement had been obtained on two occasions but the poorer one only once. If a measurement of some quantity had been found to be 1.36 when made under unfavorable circumstances, and 1.41 when made under circumstances that were more favorable to experimentation it might be considered best to assign the respective weights of 1 and 1.5 to the two values. The weighted average would then be $(2 \times 1.36 + 3 \times 1.41) \div 5$, or 1.39, a figure which will be seen to be nearer to the better value than to the poorer one in exactly the ratio of 1 to 1.5.

201. Arbitrarily Assigned Weights.—The objectionable feature of such an arbitrary assignment of weights is very obvious. The relative weights depend too much upon the judgment of the individual computer; furthermore, it is often difficult to avoid being influenced by the fact that certain determinations vary more or less widely from the expected value, instead of keeping one's judgment focussed on the quality of the experimental work.

Which do you consider the better method of determining density, by buoyancy, or by displacement? Choose what you consider the best ratio for their relative accuracies and find the corresponding weighted average, but be careful not to give extra weight to either measurement on account of its coming close to the third determination made by calculating the volume obtained by mensuration (see § 159, ¶ 3).

202. Weight and Dispersion.—Determinations of any carefully measured magnitude are usually stated in the form of an *average* and *its dispersion*, $a \pm d_{av}$. Subject to the condition that the influence of constant errors can be neglected, it can be shown mathematically that the best value for a measurement is obtained by *weighting each determination of an average in inverse proportion to the square of its dispersion.* Thus, if one determination has a dispersion of .0040 and another has a dispersion of .012 the former should be given nine times as much weight as the latter. This can be expressed in a general formula, if d is used to denote the dispersion of an average, by saying that the *weighted average* of

$$a_1 \pm d_1, \; a_2 \pm d_2, \; a_3 \pm d_3, \; \cdots$$

is

$$\frac{a_1/d_1^2 + a_2/d_2^2 + a_3/d_3^2 + \cdots}{1/d_1^2 + 1/d_2^2 + 1/d_3^2 + \cdots}$$

or

$$w.\ av. = \Sigma(a/d^2)/\Sigma(1/d^2),$$

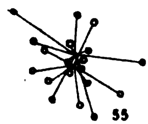
but it is much better to learn the principle involved than to memorize the formula.

203. Limitations of $\mathbf{w = k/d^2}$.—Attention should again be directed to the fact that weighting according to dispersions takes no account of the fact that constant errors may be present in the given data. The dispersion summarizes only the accidental errors, and if the constant errors are greater than these the weighted average is no better than the simple arithmetical average (Figs. 55, 56).

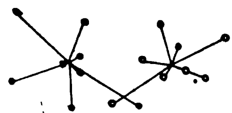
Tabulate the determinations, made by the various members of the class, of the density of aluminum as found by the effect of buoyancy. Calculate the typical value in the form $a \pm d$.

Find in the same way the average and dispersion of the density as determined by displacement. Use the slide rule, not the tables of logarithms.

Calculate the weighted average of these two data.



55



56

FIGS. 55, 56. TARGET DIAGRAMS ILLUSTRATING ERRORS.—In Fig. 55 the two groups have nearly the same center, that with the smaller dispersion naturally being the more trustworthy. In Fig. 56 at least one of the groups shows such a large constant error that the relative difference in the two sets of accidental errors is unimportant.

There are several sources of constant error in each of the two above methods of determining density. State at least four that are common to both methods, and at least one that influences one form of experiment but not the other. For example, if bubbles cling to the block when it is immersed (it is usually difficult to avoid them altogether) the apparent weight will be lessened, always causing the calculated density to be too low; they will also cause too much water to overflow, again making the calculated density lower than it should be. Consider also the effects of such things as inequality of the beam-arms of the balance, capillary attraction where the thread cuts the surface of the water, etc.

204. Exception to the Rule.—The method of weighting observations in inverse proportion to their dispersions is used for separate and independent data whose relative accuracy is assumed to be shown by their dispersions. Where two or more series of observations, however, are known to have been made with equally trustworthy apparatus, methods, and observers they should be

weighted merely according to the number of measurements which each comprises, notwithstanding that their dispersions might indicate a very different result.* To do otherwise would be to repudiate the principle of the average, which depends upon the fact that all observations are supposed to be equally trustworthy. On the other hand, when different observations are known to be unequally trustworthy, even if they occur in the same series, weight may be given to the fact that some are closely clustered about an apparent central position while others diverge erratically. A great advantage of the *median*, as a representative magnitude, is that it is not unduly influenced by a very large or a very small measurement and hence it automatically gives less weight to the more aberrant measurements of a given group.

Which is to be preferred, the average or the median, for a determination, like the one just made, of density by buoyancy? Why?

205. Questions and Exercises.—1. Why was the aluminum block hung with a thread instead of a string? Would there be any advantage in using wire instead? If wire should be used what kind of wire would be best?

2. Is it necessary to calculate dispersions in order to weight averages in accordance with § 202. What simpler calculation can be used to give exactly the same result?

3. Read § 186 again, and write in your note book a statement of the most important fact that it contains. Do not follow the form of any of the printed statements, but try to write out the fact from an essentially different point of view.

* Similarly, a single set of measurements known to be uniformly and equally good must be simply *averaged;* to give more weight to the individual measurements that have the smaller deviations would be a procedure akin to substituting the median for the average (but see also the end of § 174).

XIX. CRITERIA OF REJECTION

Apparatus.—Slide rule.

206. Observational Integrity.—When successive re-determinations of a quantity have been made in the course of an experimental investigation it is to be supposed that they have all been made with an equal degree of care.

It is important to remember that an observation should never be rejected simply because it is not in satisfactory agreement with the other determinations of the series. If the experimenter realizes that one of his measurements was made under some kind of a handicap or under such conditions that a faulty result would be likely it is permissible to cross out the corresponding value in his notes and to omit it in the final consideration of the data, but there must be some definite and satisfactory reason for discarding it other than the fact of its divergence from the expected value. The temptation, often felt by the beginner, to omit or "re-determine"* a discordant result may be very perceptible, but absolute freedom from prejudice (see *dependent measurements*, § 159; also § 47, ¶ 3) should be cultivated to such a point that the experimenter is habitually able to feel a certain disinterestedness in the outcome of a measurement after he has first taken pains to ensure its being as trustworthy as possible. His attitude should be, "I have done all that can be done in the way of preparations for making this measurement accurate and independent; now let the result turn out as it will."

* A re-determination is not intrinsically objectionable, but it should be made *in addition* to the other determination, not *in place* of it.

207. Importance of Criteria.—Even with all care to make successive measurements equally accurate it often happens that one or more of them show unduly large deviations from the average. In order to prevent these values from having an abnormal influence on the representative value certain rules have been formulated for determining whether they shall be retained or discarded, for if an observer merely used his own judgment in deciding the question the result would depend too much upon his own individuality and temperament, and different observers would obtain different results from data identically the same, just as in the case of the arbitrary assignment of weights (§ 201). In fact, the rejection of a measurement is nothing more nor less than giving it a weight equal to zero.

208. Chauvenet's Criterion.—One of the easiest to understand of the various devices for testing doubtful observations is known as *Chauvenet's criterion of rejection,* according to which rejectability is determined as a function of deviation, dispersion, and number of measurements. An unduly large deviation is an argument for rejection, especially if the dispersion is relatively small; furthermore, a deviation so large that it would not be expected to occur more than once out of a hundred cases might not seriously affect the average of a hundred values, but if it happened to occur in a set of only ten measurements it would probably exercise an altogether disproportionate effect upon their average. The object of a criterion of rejection is not to indicate that a certain measurement is wrong, but merely to point out that it is liable to be misleading it if occurs among a small group.*

* Remarkably wide deviations may be expected to occur once in a while if the number of measurements is extremely large;

The following exercise is for the purpose of showing the theory of the effect that the combined influences of deviation, dispersion, and number of measurements exert upon the determination of the advisability of keeping or rejecting any single measurement of a group.

x	y
0	50.0
1	49.9
2	49.6
3	49.0
4	48.2
5	47.2
6	46.0
7	44.6
8	43.3
9	41.6
10	39.9
12	36.0
14	32.1
16	28.0
18	24.0
20	20.2
22	16.6
24	13.8
26	10.8
28	8.4
30	6.5
32	4.8
34	3.6
36	2.6
38	1.8
40	1.2
42	0.9
44	0.7
46	0.5
48	0.3
50	0.2
52	0.1
54	0.1
56	0.0
58	0.0
60	0.0

VALUES OF $y \equiv 50e^{-(.04769x)^2}$

Draw a graphic diagram from the table. Then draw the ordinates $x = 10$ and $x = 50$ from the base line up to the curve. The result will be the right-hand half of a normal frequency diagram, the x-values corresponding to deviations and the y-values to the frequency of their occurrence. Remember that the total area of such a curve corresponds to the number of deviations (§ 164); in the same way the area between curve and base line which is bounded on the left and right by any two ordinates represents the number of observations whose numerical deviations lie between those two limits. As the dispersion is the same as the median deviation (§ 186) it is evident that the ordinate which bisects the area (this is the ordinate $x = 10$, for the scales used in *this* diagram) must have its abscissa numerically equal to the dispersion.

Suppose another ordinate is drawn at a position so far to the right that it includes between the y-axis and itself nine tenths of the total area under the curve and leaves only one tenth of the area beyond it to the right, then the cor-

notice that there is *some* space between the curve $y = e^{-x^2}$ and the x-axis even at a great distance from the y-axis (§ §67,107).

responding abscissa would similarly have a value that would be exceeded by only one tenth of the total number of deviations, and if any one deviation were chosen at random there would be only one chance in ten that it would be larger than the corresponding x-value. It can be proved mathematically* that in order to include nine tenths of the area the ordinate must be drawn 2.44 times as far to the right of the y-axis as the line which bisects the area and corresponds to the dispersion.

On your diagram draw the ordinate that includes nine tenths of the area and make sure that its abscissa fulfills the condition stated above. If the total number of measurements were ten how many would most probably be represented by the area to the right of the ordinate? How many if the number of measurements were 50? How many if the number were 4? How many if 6? The last two questions should be answered to the nearest whole number.

Since the ordinate for $x = 2.44\ d$ includes nine tenths of the area and the limit 2.44 × *dispersion* includes nine tenths of the deviations it might be said theoretically and rather figuratively that if there were only five measurements in a certain series the number of measurements whose deviations were greater than this limit would most probably be just *one half*. In other words the limit would be just on such a border line that if it were decreased we should expect it to exclude one measurement rather than no measurements, and if it were increased we should expect it to exclude no measurements rather than one measurement.

* Certain statements are intended to be taken on faith by the student. This one is just as true as the statement that the frequencies of accidental errors follow the law $y = e^{-x^2}$ or that fluid friction in a water pipe varies as the ~~1.8~~ 1.2 power of the radius; all of them can be proved but none of the proofs are necessary here.

It follows, then, that *no* one of a series of five measurements theoretically ought to have a deviation of *more* than 2.44 times the dispersion. This being the case it is only natural to consider that one is justified in discarding any one of the measurements of a series of five if its deviation does exceed this *limit.* Just as the ordinate for 2.44 d excludes one tenth of the area (*i. e.*, excludes half a measurement if there are five in all) so the ordinate at 2.57 d excludes one twelfth of the area, which would correspond to half a measurement if the total number of measurements were six. Accordingly no measurement in a set of six should theoretically have $v > 2.57\ d$ if the set is supposed to follow the law of frequency distribution for accidental errors. *Chauvenet's criterion* is simply an extension of this delimitation to other values of n as well as 5 and 6. The column, l, of the following table shows the limiting x-value for

n	l	$\log l$	n	l	$\log l$	n	l	$\log l$	n	l	$\log l$
1	1.00	000	11	2.97	473	21	3.35	525	32	3.59	554
2	1.71	233	12	3.02	480	22	3.38	529	34	3.62	556
3	2.05	312	13	3.07	487	23	3.40	532	36	3.65	561
4	2.27	356	14	3.11	493	24	3.43	535	38	3.68	566
5	2.44	387	15	3.15	498	25	3.45	538	40	3.70	570
6	2.57	410	16	3.19	504	26	3.47	540	49	3.81	581
7	2.67	427	17	3.22	508	27	3.49	543	64	3.95	597
8	2.76	441	18	3.26	513	28	3.51	546	81	4.06	608
9	2.84	453	19	3.29	517	29	3.53	549	100	4.16	619
10	2.91	464	20	3.32	521	30	3.55	551	671	5.00	699

Chauvenet's Criterion.—If the most divergent measurement out of a series of n determinations has a deviation more than l times as great as the dispersion of the individual measurements it should be rejected.

which the corresponding part of the area (see appendix) included under the curve $y = e^{-x^2}$ is $1 - 1/2n$ and the

excluded part is half of $1/n$, as before, this value being expressed in terms of the dispersion.

It should be carefully kept in mind when considering any criterion of rejection that we are interested in the individual measurements, and, accordingly, the dispersion to which the criterion applies is the *dispersion of the individual measurements*, not the dispersion of the average. Chauvenet's criterion, then, is the test of whether any deviation is greater than l times the dispersion of the individual values of a series of n measurements, where l corresponds to n in the way shown in the table.

209. The Probable Error.—By this time the student ought to be thoroughly aware of the fact that the dispersion is not properly an error, but a deviation. If he also realizes that deviations within its limits are no more probable than improbable there can be no objection to his using the term that is always employed by physicists in speaking of this characteristic deviation. In this book the term *dispersion* has been used in order to avoid repeatedly informing the student that it is an error and repeatedly suggesting that there is something very probable about it. It will hereafter be spoken of as the *probable error*, and of course it will be understood that it is used in two forms, the probable error of the individual measurements (d_1, or p_1) and the probable error of the average (d_{av}, or p_{av}).

In an experimental determination of the specific heat of lead shot the following values were obtained by a class of students. Test them by Chauvenet's criterion to determine whether any measurement falls outside of the theoretical limits, but if two or more such values are found *reject only the most divergent one*, find a

.022
.0309
.031
.032
.0347
.035
.036
.038
.045
.050

SPECIFIC HEAT OF LEAD SHOT.

new average for those that remain, and apply the criterion to them in turn.* Repeat the process, if necessary, until no more values can be discarded, and then state the best value obtainable from the figures, with its "probable error."

210. Graphic Approximation to Chauvenet's Criterion.—Where a graphic diagram is to be used for only a single series of numbers instead of for sets of values of two varying quantities it is advisable to use a horizontal scale and lay off the individual measurements as small dots or circles (Fig. 57) unless they are sufficiently numerous to allow a good histogram to be drawn.

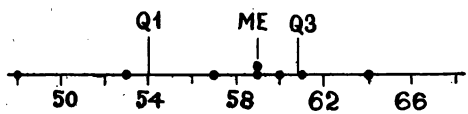


FIG. 57. DISTRIBUTION OF A FEW MEASUREMENTS.—When the majority of frequencies are very small each measurement may be represented by a dot instead of the formal square of the histogram. The centre of gravity of such a system of equal-sized dots is more readily apparent to the eye than is the position of the line that would be needed to bisect the area of the frequency polygon or histogram.

The following figures are the experimental values of the slope of the first "black-thread" diagram as obtained by a class of students: .60, .57, .64, .53, .48, .59, .59, .61. Make a graphic diagram of these values and mark the median and quartiles. Do not measure the semi-interquartile range nor multiply it by l, but mark off its length on the edge of a strip of paper and apply this to

* If one large measurement and one small one are beyond the limits the rejection of the more divergent one may result in shifting the average far enough toward the other one to bring it within bounds. Where it is obvious that this cannot happen it is unnecessary to adhere to the letter of the rule.

your diagram, laying it off to the right and to the left of the median* as many times as may be indicated by the criterion. In this way a rough application of the criterion can be made graphically and the long calculation can be avoided. Determine from the diagram whether any value should be rejected and then verify the result by the usual form of calculation.

Use the graphic method for applying Chauvenet's criterion to the following set of barometer readings:

29.986	inches
29.982	"
29.990	"
29.984	"
29.984	"
29.980	"
29.986	"
29.977	"
29.984	"
29.982	"
29.986	"
29.988	"
29.984	"

BAROMETER READINGS.

What advantage has the arithmetical method over the graphic method?

Write down, in your own words, just what it is that is represented by (*a*) the probable error of a single measurement, $.6745\sqrt{\Sigma d^2/(n-1)}$, and by (*b*) the probable error of the average, $.6745\sqrt{\Sigma d^2/n(n-1)}$.

211. Irregularities of Small Groups.—The probable error, or "dispersion," cannot be considered as having much meaning in cases where the total number of measurements is less than ten, and even with ten measurements it should be treated with a certain amount of caution. A number of values less than ten will hardly ever give a histogram of their frequency-distribution which is recognizably similar to the graph of $y = e^{-x^2}$, the curve which all unbiassed measurements will be found to follow if they are sufficiently numerous.

212. Justification of the Criterion.—For the same reason, it is hardly worth while to use a criterion of rejection for less than ten measurements. The example given above with only five is intended merely for an illustration of the method of using the criterion, and the

* It is better to use the mid-quartile point than the median in case the two are not the same.

still smaller values in the table are only of theoretical importance. Chauvenet's criterion is not to be considered as showing that any one measurement is a mistake, but only as indicating that a very large deviation is such a rarity that it would have an unduly large influence upon the average if it were allowed to remain along with the other values of a very limited series of measurements (§ 208, page 231, foot-note).

213. Wright's Criterion.—Another criterion of rejection, which is sometimes employed, is that of Wright. According to this *the arbitrary rejection of a single measurement may be considered if its deviation is more than five times the probable error.*

Turn back to the graphic diagram of the table in § 208, and notice how small a part of the area of the curve lies to the right of the ordinate, $x = 50$, which represents five times the probable error. Turn to the table of values for Chauvenet's criterion and note how many measurements would need to be made before "half a measurement" would be likely to diverge from the average five times as far as the probable error.

In the measurements to which you have already applied Chauvenet's criterion how many would have been rejected if Wright's criterion had been used instead? If a deviation is great enough to be practically sure of rejection by one of the two criteria will it ordinarily be rejected by the other? If a maximum deviation is small enough to avoid rejection by one criterion will it be practically certain to escape rejection by the other? Explain why. What are the relative advantages of the two criteria?

214. Comparison of Characteristic Deviations.—Other limiting values, which give practically the same result as Wright's criterion, are *four times the average deviation*, and *three times the standard deviation.*

Turn back to your notes on the use of logarithms and find the graphic diagram of $y = e^{-x^2}$. Mark off the following values on the base line, $p = .4769363$, $a = .5641895$ and $s = .7071066$. These represent respectively the probable error, the average deviation, and the standard deviation, and are roughly proportional to 10 : 12 : 15; a better approximation to their ratios than is given by 10 : 12 : 15 may be found with the aid of the slide rule. Draw the corresponding ordinates, and notice that the last one meets the curve at the *point of inflection,* that is, at the point where it is momentarily straight as it changes from convex upward to convex downward.

215. Questions and Exercises.—1. Write a definition of Chauvenet's criterion in your own words. Notice that the last sentence of § 208 is not a satisfactory definition.

2. Can a set of three measurements comprise such values that one of them will be rejected by Chauvenet's criterion? Give an example to illustrate your answer.

3. Can a set of two measurements comprise such values that one of them will be rejected by Chauvenet's criterion. Give an example to illustrate your answer.

XX. LEAST SQUARES

Apparatus.—Slide rule; black thread.

216. The Average as a Least-Square Magnitude.—The mathematical *principle of least squares* is that when measurements are equally trustworthy their best representative value is that for which the sum of the squares of the deviations has the lowest numerical value.* It is upon this principle that the use of the average is based, for it is easy to show that the sum of the squares of the deviations of any particular set of numbers will be greater when measured from some other value than when measured from the average.

Find the average of the numbers 3, 3, 4, 5, 10; also their deviations from the average, and the sum of the squares of the deviations. Find the median of 3, 3, 4, 5, 10; and the sum of the squares of the deviations from the median. Find the sum of the squares of the deviations from the harmonic mean (call it 4.1) or from the mode, and notice that $\Sigma(v^2)$ is smaller when the deviations are measured from the average than when measured from any of the other numerical values.

217. Least Squares for Conditioned Measurements.—If we are dealing with two conditioned measurements, as in the case of the x-values and the y-values of the black-thread experiment, the principle of least squares

* This principle can be proved from the "fact of experience" that deviations follow the law of the exponential equation $y = e^{-x^2}$. As an example of its application consider the marks 6, 8, and 7, on a scale of ten, which one student obtained on three examination questions, and the marks 7, 7, 7, which were obtained by another student. Find the deviations from the theoretical mark 10, and see which student has the smaller value for $\Sigma(v^2)$.

shows that the line which expresses the *condition* or gives the law of relationship between the two variables must be so placed that the sum of the squares of the distances from it to all of the experimental points shall have the smallest possible value.

The x and y of any one of the points cannot in general be substituted in the black-thread equation, $y = a + bx$, but $a + bx - y$ will have some small positive or negative value instead of being equal to zero. If the various points are considered to have the definite positions denoted by (x_1, y_1), (x_2, y_2), (x_3, y_3), etc., it will usually be found that none of these sets of values will exactly satisfy the equation $y = a + bx$ or $a + bx - y = 0$, but will give such a result as $a + bx_1 - y_1 = d_1$, where d is some small quantity whose exact value need not be determined; similarly, the other points will give other equations, $a + bx_2 - y_2 = d_2$, $a + bx_3 - y_3 = d_3$, etc., and according to the principle of least squares the sum $d_1^2 + d_2^2 + d_3^2 + \cdots$, must be as small as possible.*

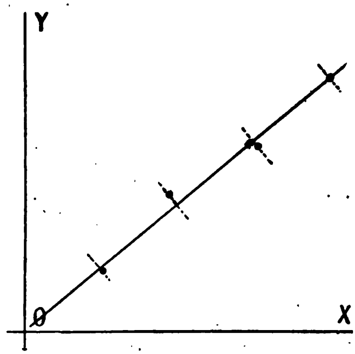


FIG. 58. THE PRINCIPLE OF LEAST SQUARES.—The general principle is that the theoretical relationship is to be so arranged that the sum of the squares of the discrepancies of the actual measurements shall be as small as possible. If a straight line is to be used it must be so arranged that the sum of the squares of the distances to it from the various points is a minimum.

* It can be shown mathematically that the distance from the point (x_1, y_1) to the line $y = a + bx$ is proportional to $a + bx_1 - y_1$.

This means that the sum of the squares of the left-hand members of the equations, or $\Sigma(a + bx_n - y_n)^2$ must have its minimum value, and it can be shown by processes of pure mathematics that this will be the case if

$$a = \frac{\Sigma(x)\Sigma(xy) - \Sigma(y)\Sigma(x^2)}{(\Sigma x)^2 - n\Sigma(x^2)}$$

and

$$b = \frac{\Sigma(x)\Sigma(y) - n\Sigma(xy)}{(\Sigma x)^2 - n\Sigma(x^2)},$$

where x and y stand for $x_1, x_2, x_3, \cdots$ and $y_1, y_2, y_3, \cdots$ the x-values and y-values of the experimental points.

These equations enable one to determine the values of a and b, and hence to find the position (see § 105) that a straight line (black thread) must have if it is to be so located that the sum of the squares of the distances to it from all of the experimental points shall have the smallest possible value.

By similar processes a, b, and c could be found for the equation of the parabola $y = a + bx + cx^2$, or the appropriate coefficients for curves having even more complicated equations, but the processes of computation become so tedious that it is better to replace the variables, as explained in the lesson on graphic analysis, by others that will conform to the straight line law. The diagram (Fig. 58) shows some of the points that correspond to the table in § 118, for which the position of the line through the origin was found graphically by the use of a black thread.

Tabulate the values of x and y for the black-thread experiment of § 115, arranging them as shown in the following table, and writing the proper numerical values in the spaces marked Σx, Σy, $\Sigma(xy)$, and $\Sigma(x^2)$. Then calculate the values of a and b from the formulæ, arrang-

ing the work neatly and being careful to avoid using the wrong algebraical signs or confusing $\Sigma(x^2)$ with $(\Sigma x)^2$. Keep only three significant figures in the final results.

Write the equation representing the best position of the black thread in the form $y = a + bx$, and then in the form $x/m + y/n = 1$. Compare the calculated values of the intercepts with your experimental values that were obtained in the work on Graphic Analysis (chap. x).

x	y	xy	x^2
1	9.8	9.8	1
2	8.5	17.0	4
3	8.0	24.0	9
4	7.2	28.8	16
5	6.7	33.5	
6	6.5		
7	6.2		
8	5.5		
9	5.0		
10	4.1		
11	3.9		
12	3.2		
13	2.3		
Σx	Σy	$\Sigma(xy)$	$\Sigma(x^2)$

METHOD OF LEAST SQUARES FOR A LINEAR LAW.

218. Least Squares and Proportionality.—If two variables, x and y, are always proportional the linear equation $y = a + bx$ reduces to the form

$$y = 0 + bx$$

and the best value of the coefficient will be found to be

$$b = \frac{\Sigma(xy)}{\Sigma(x^2)}.$$

Let $a = 0$ in the equation

$$a = \frac{\Sigma(x)\Sigma(xy) - \Sigma(y)\Sigma(x^2)}{(\Sigma x)^2 - n\Sigma(x^2)};$$

then

$$\Sigma(x)\Sigma(xy) = \Sigma(y)\Sigma(x^2).$$

Multiplying by $n\Sigma(x)$,

$$n(\Sigma x)^2\Sigma(xy) = n\Sigma(x)\Sigma(y)\Sigma(x^2);$$

whence

$$\Sigma(x)\Sigma(y) : (\Sigma x)^2 :: n\Sigma(xy) : n\Sigma(x^2).$$

But if $a : b :: c : d$, then $a - c : b - d :: c : d$; accordingly,

$$\frac{\Sigma(x)\Sigma(y) - n\Sigma(xy)}{(\Sigma x)^2 - n\Sigma(x^2)} = \frac{n\Sigma(xy)}{n\Sigma(x^2)},$$

or

$$b = \Sigma(xy)/\Sigma(x^2).$$

In studying the density of water (§ 118) the ratio of $\sqrt{y}$ to x was found graphically to be about 0.83. Determine the accurate value of this ratio by calculating each product of the variables x and $\sqrt{y}$, summating, and dividing by the summated squares of the x's.

219. Least Squares for a Theoretically Constant Value.—If the variation of y is supposed to be *nil, i. e.*, if y is a constant, the best representative value for the fluctuating experimental determinations of it can easily be found by the method of least squares:

In the equation $y = a + bx$ if $b = 0$ the equation

$$b = \frac{\Sigma(x)\Sigma(y) - n\Sigma(xy)}{(\Sigma x)^2 - n\Sigma(x^2)}$$

gives

$$\Sigma(x)\Sigma(y) = n\Sigma(xy).$$

Eliminating $\Sigma(xy)$ between this equation and the equation

$$a = \frac{\Sigma(x)\Sigma(xy) - \Sigma(y)\Sigma(x^2)}{(\Sigma x)^2 - n\Sigma(x^2)}$$

gives

$$a = \frac{(\Sigma x)^2\Sigma(y)/n - \Sigma(y)\Sigma(x^2)}{(\Sigma x)^2 - n\Sigma(x^2)},$$

which is obviously the same as

$$a = \Sigma(y)/n.$$

In other words, the best representative value of y is obtained by adding all n of its experimental values and dividing by n,—as already stated without proof.*

220. Consecutive Equal Intervals.—If the successive unknown intervals of a scale are not perceptibly different from one another their value can easily be found by the method of coincidences, as illustrated in Fig. 48, § 153. If the scale is a crude one or the method of measuring is a precise one the intervals which ought to be equal will be found to differ among themselves (§ 160) and the question arises as to the best representative value for them. The principle, of course, is to find the perfectly uniform scale whose intervals are of such size as to make the sum of the squares of the discrepancies in the graduations of the irregular scale as small as possible (Fig. 59). It will not do to take the average of the lengths of all the consecutive intervals because the result that would be obtained would depend only upon the position of the first graduation and the last graduation, and the data afforded by all the rest of the graduations would have been neglected.

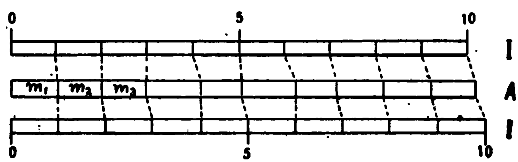


FIG. 59. BEST VALUE FOR CONSECUTIVE INTERVALS.—An imaginary uniform scale is to be so chosen that $\Sigma(v^2)$ for the irregular intervals is a minimum. *A*, actual scale; *I*, *I*, imaginary scales.

* Remember that the principle of least squares is based on the assumption that all of the measurements under consideration are equally trustworthy. If this condition could always be fulfilled we should have little use for any representative value except the average (*cf.* §§ 174; 204).

If the distance from the zero of the scale to the *graduation that is marked* x is called y the problem is to find the best value of b in the equation $y = bx$ (Fig. 60). The formula $b = \Sigma(xy)/\Sigma(x^2)$ of § 218 cannot be used in this case, for the experimental deviations cannot involve both x and y, but must be limited to the y-values on account of the necessity of keeping the difference between successive x-values constant. It is not the squares of the *perpendicular* distances in Fig. 58 whose sum must be a minimum, but the squares of the *vertical* distances in Fig. 60. The formula that is commonly used in such cases can be obtained as follows:

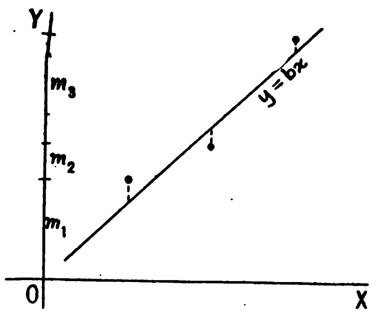


Fig. 60. Least-Square Adjustment for Equal Intervals.

If the n x-values of the n points in Fig. 60 are $x_1 = 0$, $x_2 = 1$, $x_3 = 2$, $\cdots$, $x_n = n - 1$, the slope b which will give the best value, m, of the $n - 1$ intervals, m_1, m_2, m_3, $\cdots$, can be calculated as follows:

Since

$$\Sigma x = 0 + 1 + 2 + 3 \cdots + n = \tfrac{1}{2}n(n + 1)$$

and

$$\Sigma(x^2) = 0^2 + 1^2 + 2^2 + \cdots + n^2 = \tfrac{1}{6}n(n + 1)(2n + 1)^*,$$

the formula

$$b = \frac{\Sigma(x)\Sigma(y) - n\Sigma(xy)}{(\Sigma x)^2 - n\Sigma(x^2)}$$

* The correctness of these formulæ may be taken for granted. They are easily proved by the processes of elementary algebra.

becomes

$$b = \frac{\frac{1}{2}n(n+1)(y_1 + y_2 + \cdots y_n) - n(y_1 + 2y_2 + 3y_3 + \cdots + ny_n)}{\frac{1}{4}n^2(n+1)^2 - \frac{1}{6}n^2(n+1)(2n+1)}.$$

Multiplying both numerator and denominator by $12/n$ gives

$$b = 6\,\frac{(n+1)(y_1 + y_2 + \cdots + y_n) - 2(y_1 + 2y_2 + \cdots + ny_n)}{3n(n+1)^2 - 2n(n+1)(2n+1)}.$$

Simplifying the denominator and multiplying the fraction by $-1/-1$ gives

$$b = 6\,\frac{2(y_1 + 2y_2 + 3y_3 + \cdots + ny_n) - (n+1)(y_1 + y_2 + \cdots + y_n)}{n^3 - n}.$$

Collecting the coefficients of the y's,

$$b = 6\,\frac{(1-n)y_1 + (3-n)y_2 + (5-n)y_3 + \cdots + (n-3)y_{n-1} + (n-1)y_n}{n(n^2-1)}.$$

Finally, collecting the coefficients of $(n-1)$, $(n-3)$, etc., gives

$$6\,\frac{(n-1)(y_n - y_1) + (n-3)(y_{n-1} - y_2) + (n-5)(y_{n-2} - y_3) + \cdots}{n(n^2-1)}$$

for the best value of the interval. The point x_1 corresponds, of course, to the zero of the scale and y_1 is

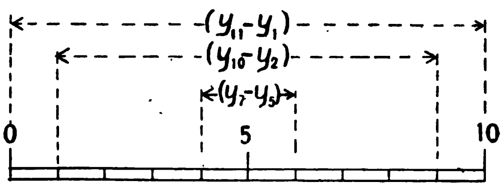


FIG. 61. TREATMENT OF UNIFORM INTERVALS.—The best representative value, according to the formula, is $10(y_{11} - y_1) + 8(y_{10} - y_2) + 6(y_9 - y_3) + 4(y_8 - y_4) + 2(y_7 - y_5) + 0(y_6 - y_6)$, multiplied by $6/n(n^2 - 1)$. Notice that an *even* number of intervals causes the middle graduation to be neglected.

numerically equal to zero.* It should be noticed that when there are an even number of intervals the middle point does not affect the formula. Accordingly it is preferable to use an odd number of intervals, unless there is some reason to the contrary.

The following numbers were obtained by measuring the consecutive intervals between eleven parallel lines that were intended to be located one tenth of a millimetre apart on a glass slide used for microscopic measurements. The numbers give the corresponding distances in units of an arbitrary scale located in the eye-piece of the microscope so as to appear superimposed on the image of the object under examination. Find the best representative value for these ten subdivisions; then note which subdivision (if any) agrees in length with the determination.

subdivision	*length*
1st	160
2d	163
3d	164
4th	166
5th	165
6th	159
7th	162
8th	166
9th	165
10th	166

EXAMPLE OF SCALE INTERVALS.

The same method can be used for equal changes of length that are due to any uniform cause; for example, a spring elongates by uniform intervals when weights proportional to 1, 2, 3, 4, ⋯ are hung on it. It is also applicable to intervals of time, *e. g.*, the period of a pendulum or of the indicating pointer of a galvanometer or analytical balance; and, indeed, to any variable that is proportional to another quantity which can be varied by equal intervals; thus, the π-disc (§ 49) may be rolled until any given point on its circumference has touched the flat scale at several equidistant points, or the area

* The student should be careful to avoid the common mistake of using y_2 instead of y_1 for the first interval of the scale (Fig. 61). The result would be that only the other $n - 2$ intervals would be available.

indicated by a planimeter may be read when the tracing point has been carried around the periphery once, twice, thrice, etc.

221. Equal Intervals without Least Squares.—The black-thread method can of course be used for measurements like those that have just been considered, but if the uncertainty of judgment that is necessarily associated with it is objectionable a simple calculation can be performed which is free from the disadvantage of averaging stated in the first paragraph of § 220. The distance from y_1 to $y_{(n+1)/2}$ or $y_{(n+2)/2}$ is measured, also, from y_2 to $y_{(n+3)/2}$ or $y_{(n+4)/2}$, etc., and each of the measurements is divided by $(n - 1)/2$ or $n/2$, the quotients being finally averaged. Thus, for the scale shown in Fig. 59, one fifth of the distance from the mark 0 to the five-inch mark is averaged with one fifth of the distance from 2 to 6, one fifth of 3 to 7, etc., so that all of the graduation marks are utilized.*

222. Simultaneous Indirect Measurements.—In certain kinds of measurement several quantities (usually either two or three) have to be determined by methods which will not allow each to be measured separately but which furnish functional relations in which there is always more than one unknown involved. For example, two magnets may have the strengths of their poles determined by measuring their relative intensities as a ratio and their mutual attraction as a product. The data $x/y = a$ and $xy = b$ are then sufficient to determine both x and y. Sometimes only one unknown

* The procedures given in § 220 and § 221 are the ones that are ordinarily used for physical measurements. Whether their results are preferable to those of the more general methods stated previously will be an interesting question for the student to determine for himself. He should make up at least one example of an extremely irregular set of scale-divisions.

quantity needs to be measured, but cannot be determined except along with a different one. For example, by the use of an ammeter to measure the flow of electric current that a particular battery can drive through an unknown resistance it is easy to deduce the numerical value of the resistance, but the method must be modified before it can be used to measure the resistance of a "ground," *i. e.*, of the place where the current passes from the negligibly resistant wire into the body of the non-resistant earth, for the current must return from the earth to the battery through some other resistant "ground." When there are two such points, it is possible to find the sum of their resistances but not the value of either one alone. If three separate connections to earth are available, their resistances may be called x, y, and z, and after measuring in turn the resistances $x + y$, $y + z$, and $x + z$ the value of any one of them may be found by solving the three simultaneous equations that are obtained. If there are four such points, however, more separate and independent observations are obtainable than there are unknown quantities, and the determinations will need adjustment by the method of least squares. If the resistances are called w, x, y, and z, then the following data are available:

$$\left.\begin{array}{llllllll} w & + & x & & & & = a \\ & & x & + & y & & = b \\ & & & & y & + z & = c \\ w & & & + & y & & = d \\ & & x & & & + z & = e \\ w & & & & & + z & = f \end{array}\right\}$$

In general no exact solution will be possible, and the problem is to determine such a solution that the sum of the squares of the deviations of the observed values from

the *adjusted values* that are given by the assumed solution shall be as small as possible.

A graphic illustration of the effect of more equations than unknowns is given in Fig. 62, which represents a set of experimental determinations

$$\left.\begin{aligned} 3x - y &= 3 \\ y - x &= 2 \\ 2x - y &= 0 \end{aligned}\right\}.$$

By plotting these three equations on one diagram it will be seen that the three straight lines do not pass through a single common intersection, but a point (2.4, 4.9) can be found which comes fairly close to satisfying the three conditions.

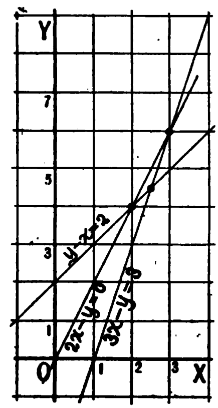


FIG. 62. THREE OBSERVATIONAL EQUATIONS.—Since all observations contain errors the three equations are not consistent, but the best point to represent the intersection of the three lines can be found by choosing it so that the sum of the squares of the discrepancies is a minimum.

The general method of obtaining the best representative values of the unknown quantities can be shown by the application of the principles of the differential calculus to be as follows:

1. Multiply each equation by the coefficient of x in that equation, and add the results. The sum is called a *normal equation*.

The equations of Fig. 62,

$$\begin{aligned} 3x - y &= 3 \\ -x + y &= 2 \\ 2x - y &= 0, \end{aligned}$$

thus give

$$\begin{aligned} 9x - 3y &= 9 \\ x - y &= -2 \\ 4x - 2y &= 0 \\ \hline 14x - 6y &= 7. \end{aligned}$$

2. Multiply each equation by the coefficient of y in that equation, and add the results in order to obtain a second normal equation:

$$\begin{aligned} -3x + y &= -3 \\ -x + y &= 2 \\ -2x + y &= 0 \\ \hline -6x + 3y &= -1. \end{aligned}$$

3. If the equations contain third, fourth, fifth, $\cdots$ nth unknown quantities, form a normal equation for each one in the same way, multiplying each equation by its proper coefficient of the particular unknown quantity and adding, thus obtaining n normal equations in n unknown quantities.

4. Solve the normal equations by any algebraical process. The result will be the least-square values of the unknowns. In the example under consideration this gives

$$\begin{aligned} x &= 5/2, \\ y &= 14/3. \end{aligned}$$

Choose any other point that you think best and show that it gives a larger value for $\Sigma(v^2)$ than this one does. Write the equations in the form

$$\begin{aligned} 3x - y - 3 &= v_1, \\ y - x - 2 &= v_2, \\ 2x - y &= v_3. \end{aligned}$$

Solve the following equations by the method of least squares:

$$\begin{aligned} x + 2.65y - 0.33z &= 6.21, \\ x + 2.37y + 0.12z &= 3.18, \\ x + 2.25y - 0.29z &= 5.89, \\ x - 0.87y + 3.27z &= 4.28, \\ x + 3.38y + z &= 4.07. \end{aligned}$$

Be careful that each product is given the right sign; check them as each equation is completed. If squared paper is not used be careful to keep the decimal points under one another. The tedium of adding positive and negative decimals can be greatly relieved by *overlining each negative digit** and adding *en masse.*

An *approximate method* of obtaining normal equations is sometimes used in order to avoid the increased labor of applying the above processes, 1, 2, 3, 4, to observation equations that contain decimal fractions, but it is often unsatisfactory: Make the coefficient of x positive in each of the n equations by multiplying through by -1 where necessary; then add the resultant n equations to form the first normal equation. Make the coefficients of y positive in the same way, and add the n equations to obtain a second normal equation. Proceed in this way until n normal equations in n unknown quantities have been formed; then solve. It is fairly obvious that this method will be the most satisfactory in cases where all the coefficients of all the observation equations are of approximately the same size.

* Find the value of $2.4123 - 3.6418 + 5.7827$ column by column as follows:

$$\begin{array}{r} 2.4123 \\ \bar{3}.\bar{6}\bar{4}\bar{1}\bar{8} \\ 5.7827 \\ \hline +\,4.5532 \end{array}$$

Verify these also:

$$\begin{array}{r} 3.48 \\ \bar{7}.\bar{2}\bar{2} \\ 6.35 \\ \bar{1}.\bar{8}\bar{7} \\ \hline +\,0.74 \end{array} \qquad \begin{array}{r} .\bar{8}\bar{7}\bar{4} \\ .454 \\ .\bar{6}\bar{5}\bar{3} \\ \bar{2}.\bar{8}\bar{4}\bar{0} \\ 3.380 \\ \hline 0.\bar{5}\bar{5}\bar{3} \\ = -\,.553 \end{array}$$

Apply the approximate method to the last exercise and compare the ease of computation and the accuracy of solution.

Apply the approximate method to the equations of Fig. 62, and note one objectionable feature which it may have.

To what extent are *graphic methods* applicable to the solution of simultaneous indirect measurements?

Note to §217: It has recently been pointed out by H. M. Roeser (Physical Review, Jan., 1917) that the point $x_m\ y_m$ satisfies the equation

$$y = \frac{\Sigma x \Sigma xy - \Sigma y \Sigma x^2}{(\Sigma x)^2 - n\Sigma x^2} + \frac{\Sigma x \Sigma y - n\Sigma xy}{(\Sigma x)^2 - n\Sigma x^2}\, x$$

if x_m and y_m are the average abscissa and average ordinate of all of the points; the black thread can then be placed with greater certainty, as it needs merely rotation about this point.

XXI. INDIRECT MEASUREMENTS

Apparatus.—Slide rule.

223. Importance of Indirect Measurements.—An indirect measurement is one that is calculated from one or more direct measurements instead of being directly observed by the experimenter.

In a certain sense almost all measurements are indirect, especially those that require the highest degrees of accuracy. In making a careful determination of the weight of an object the direct determinations are of the turning points of the pointer that swings back and forth as the beam of the balance oscillates. From these points the resting position of equilibrium is calculated, and from this position and a similar one obtained when the balance is empty a calculation is made of the discrepancy between the weight of the object and that of the standard weights that are balanced against it, using a previous determination of the shift of resting point that is caused by a very small standard weight. If the process of weighing is for the purpose of obtaining *mass* instead of *weight* still further calculating is necessary.

The term, *indirect*, however, is commonly applied to measurements like that of the density obtained by dividing mass by volume, one or more carefully determined measurements being treated by calculation in such a way as to give a required "indirect" result.

In some cases values calculated from other data are a necessity. The density of a solid may be obtained by direct measurement if it can be tested by immersing it in fluids whose densities are known, and this method is found to be very convenient for testing small objects

that do not have a very great density, such as precious stones.* For denser objects, however, some indirect method is necessary. The usual one is to calculate $d = m/v$, v also being obtained indirectly from the loss of weight on immersion in a fluid of known density.

In some cases much time and laborious calculation can be saved by calculating the average and probable error, $a \pm d_{av}$, for each quantity that is measured directly and then investigating the resultant probable error of the indirect measurement in accordance with theoretical considerations. For example, it would take much longer to calculate a density from each one of a set of twenty-five determinations of mass and volume and then average the results than it would to average the masses and the volumes separately and perform a single division. When using the latter method it is possible to find what the probable error of the density will amount to by investigating the probable errors of mass and volume.

In a case like the one just mentioned it would also be possible to make twenty-five calculations of density from the twenty-five sets of measurements of mass and density and to compute the probable error of the average density. In many cases, however, such a process cannot be carried out. If there were more data at hand for the mass than for the volume the excess could not be utilized; and, furthermore, some of the values needed in the formula may be predetermined constants, such as those mentioned in chapter xviii (see § 197), which are given only in the form of a representative magnitude and its probable error. The question then arises as to

* A solution of mercuric potassium iodide, *sp. gr.* = 3.11 or less, is used by jewelers for this purpose. It is stated that densities as high as 3.56 can be determined by using the double iodide of barium and mercury.

the way in which the probable error of the indirect measurement is influenced by the size of the probable errors of the direct measurements on which it is based.

224. Probable Error of a Sum.—The simplest case is when the indirect measurement is merely the sum of the two independent direct measurements. Let the direct measurements be denoted by the small letters,

$$a_1 \pm p_1 \quad \text{and} \quad a_2 \pm p_2,$$

when stated in the form of average and probable error, and let the indirect measurement with its probable error be represented by capitals,

$$\mathrm{A} \pm \mathrm{P},$$

the direct measurement being equal to the sum of the indirect measurements, so that $\mathrm{A} = a_1 + a_2$; then it can be proved that

$$\mathrm{P}^2 = p_1^2 + p_2^2.$$

The dispersion of the sum is not as large as the sum of the two direct dispersions from which it is calculated. This is because positive and negative deviations will counterbalance each other to a certain extent. It is larger than either of the others alone, however, for the extra measurement gives an extra degree of uncertainty.

A bench has a height of 42.50 ± .03 cm. above the floor, and a table is 44.35 ± .04 cm. higher than the bench. Write the height of the table, with its probable error.

225. Probable Error of a Difference.—If the indirect measurement, A, is equal to the difference, $a_1 - a_2$, between two direct measurements, a_1 and a_2, it is perhaps a natural inference that the square of its dispersion, P^2,

should be equal to $p_1^2 - p_2^2$. This is not the case, however, but

$$P^2 = p_1^2 + p_2^2$$

in all cases in which

$$A = a_1 \pm a_2.$$

The table that was measured in the last exercise is 51.10 ± .04 cm. lower than a shelf which is 137.95 ± .03 cm. above the floor. How high is the table?

Notice that the same formula applies to both of these exercises. The difference of two measurements has as large a probable error as their sum. Measuring first up and then down does not give a more precise result than measuring up and then further up.

226. Probable Error of a Multiple.—If

$$A = c_1 a_1,$$

where c_1 is some constant not subject to error, then

$$P^2 = c_1^2 p_1^2, \quad \text{or} \quad P = c_1 p_1.$$

The diameter of a circular disc is found to be 7.98 ± .03 cm. What is its circumference?

Notice that the *relative* dispersion of the circumference (3/800) is the same as the relative dispersion of the diameter (*vide infra*).

227. Associative Law:—

In general, if

$$\mathbf{A} = \mathbf{c}_1\mathbf{a}_1 \pm \mathbf{c}_2\mathbf{a}_2 \pm \mathbf{c}_3\mathbf{a}_3 \pm \cdots$$

then

$$\mathbf{P}^2 = \mathbf{c}_1^2\mathbf{p}_1^2 + \mathbf{c}_2^2\mathbf{p}_2^2 + \mathbf{c}_3^2\mathbf{p}_3^2 + \cdots \tag{1}$$

where p_n is the probable error of the average a_n.

This formula can be used to find the probable error of an algebraical expression when the probable error of each of its terms is known.

A wall consists of 15 courses of bricks, each of which is 56.5 ± .5 mm. thick, separated by 14 layers of mortar which have an average thickness of 7.5 ± 1.5 mm. Show that the probable error of the height of the wall is ± 22.3 mm., and state how much this figure would be reduced if the bricks were absolutely uniform. How much if the mortar was of uniform thickness instead?

228. Probable Error of a Product.—If two independent measurements are multiplied together the probable error of the product will follow the law expressed in the following equation.

If

$$A = a_1a_2$$

then

$$P^2 = p_1^2a_2^2 + a_1^2p_2^2.$$

Likewise if

$$A = a_1a_2a_3$$

then

$$P^2 = (p_1a_2a_3)^2 + (a_1p_2a_3)^2 + (a_1a_2p_3)^2,$$

and similarly for any number of factors; but it is more satisfactory in practice to make use of the *relative* probable error (relative dispersion, § 195) as in the following form, which is easily deducible from the form just given.

If

$$A = a_1\, a_2\, a_3 \cdots$$

then

$$(P/A)^2 = (p_1/a_1)^2 + (p_2/a_2)^2 + (p_3/a_3)^2 + \cdots.$$

Prove that the last equation (as far as it goes) is equivalent to

$$P^2 = (p_1a_2a_3)^2 + (a_1p_2a_3)^2 + (a_1a_2p_3)^2.$$

A rectangular block measures 20.00 ± .04 cm. in length, 10.00 ± .01 cm. in breadth, and 5.00 ± .01 cm. in thickness. What is its volume?

229. Probable Error of a Power.—If

$$A = a_1^n$$

then

$$(P/A)^2 = (n^2p_1^2/a_1^2) \quad \text{or} \quad P/A = np_1/a_1;$$

and, likewise, if

$$A = c_1a_1^n$$

then

$$P/A = np_1/a_1,$$

the constant not appearing in the formula if the relative probable error is used.

In the particular case for which $n = 1$, notice that the *relative* probable error of c_1a_1 is the same as the relative probable error of a_1 itself (*vide supra*).

One edge of a cubical block is $10.00 \pm .01$ cm. What is its volume? What is the area of its total surface?

Does the rectangular block (above) appear to have been measured as accurately as the cubical block? How do their volumes compare in respect to accuracy? What is there about the data of the rectangular block which make its volume determinable as closely as that of the cubical block, although its measurements are less precise?

Show that the combined volume of the two blocks is $2000. \pm 4.2$ cm^3.

230. Distributive Law:—

In general, whether the exponents are positive, negative, or fractional, if

$$\mathbf{A = c_1a_1^m a_2^n a_3^r \cdots}$$

then

$$\mathbf{(P/A)^2 = (mp_1/a_1)^2 + (np_2/a_2)^2 + (rp_3/a_3)^2 + \cdots.} \quad (2)$$

This formula can be used to find the probable error of any single term of an algebraical expression when the probable errors of its factors are known.

The formula for the volume of a cylinder is

$$v = \pi l r^2.$$

If the measurements of l and r have respective probable errors of p_l and p_r find the value of p_v, the probable error of the calculated volume.

Test the accuracy of the italicized statement (that $d_{av} = d_1/\sqrt{n}$) of § 193 by letting p denote the probable error of each of the n single measurements and finding the probable error of their average.

231. Recapitulation.—In the following formulæ the relative dispersion (P/A, or p/a) is represented by R or r.

If	*Then*
$A = a_1 \pm a_2 \pm \cdots$	$P^2 = p_1^2 + p_2^2 + \cdots$
$A = c_1a_1$	$P = c_1p_1$
$A = c_1a_1 \pm c_2a_2 \pm \cdots$	$P^2 = (c_1p_1)^2 + (c_2p_2)^2 + \cdots$

PROBABLE ERRORS OF INDIRECT MEASUREMENTS.—In these cases the probable error (p) for each average (a) is most convenient for calculation. The capital letters refer to the indirect measurement.

If	*Then*
$A = a_1a_2a_3 \cdots$	$R^2 = r_1^2 + r_2^2 + r_3^2 + \cdots$
$A = a_1^n$	$R = nr$
$A = c_1a_1^n$	$R = nr$
$A = c_1a_1^m a_2^n a_3^s \cdots$	$R^2 = (mr_1)^2 + (nr_2)^2 + (sr_3)^2 + \cdots$

PROBABLE ERRORS OF INDIRECT MEASUREMENTS.—In these cases the use of the *relative* probable error (r) simplifies the calculation.

Notice the complete formal correspondence between (*absolute*) *probable errors* after adding, multiplying by a coefficient, and assembling terms, and *relative probable errors* after multiplying, raising to a power, and assembling factors, respectively.

A bowl whose interior is an exact segment of a sphere is found to have a depth of 25.00 ± .02 centimeters and a diameter across the top of 50.00 ± .30 centimeters. Find its capacity from the formula for the volume of a spherical segment, $v = \pi hr^2/2 + \pi h^3/6$, where h is the height or depth of the segment and r is the radius of its circular base; find the probable error of the capacity by applying the second general equation to each term of the formula and then using the first general equation to determine the final result. Notice the relative probable error of the radius, r, is the same as that of the diameter, d. Arrange the calculation systematically in order to avoid numerical mistakes, and if there is any trouble in making the substitution write out each step of the process; for example:

$a_1 = 25.$	$P^2/A^2 = 3^2(.02)^2/25^2$	$\log 25 =$	1.3979
$p_1 = .02$	$P^2 = 3^2(.02)^2\pi^2 25^6/25^2 6^2$	$\log \pi^2$	0.9943
$m = 3$	$= \pi^2(.01)^2 25^4$		$\bar{4}.0000$
$c_1 = \pi/6$			5.5916
$A = \pi h^3/6$			2.5859
		$\log P =$	

232. Graphs of Propagated Errors.—It has been seen that the probable errors of two or more direct measurements are propagated through any kind of a calculation and give the indirect measurement a probable error whose formula is of the type $\sqrt{x^2 + y^2}$. Since the square root of the sum of two squares can always be represented by the hypothenuse of a right-angled triangle a graphic solution of the probable error of an indirect measurement is easily effected.

From any "origin" draw a horizontal line and a vertical line, making their lengths equal to the dispersions 3 and 4 of the measurements in § 225. Complete

the right-angled triangle and note that the length of the hypothenuse, 5, is the "propagated" dispersion of the indirect measurement, as previously found by calculation.*

The fact that $p_1^2 + p_2^2 + p_3^2$ is the same thing as $\sqrt{p_1^2 + p_2^2}^2 + p_3^2$, namely, the sum of *two* squares, makes this method extensible to any number of terms (Fig. 63).

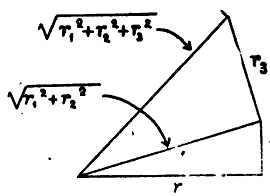


FIG. 63. GEOMETRIC DETERMINATION OF PROPAGATED ERRORS.

Use the geometrical construction to find the square root of the sum of the squares of 4, 3, and 12. Afterward, verify your result by calculation.

233. Relative Importance of Compound Errors.—The fact that an indirect probable error which depends upon the measurement of two or more different quantities always assumes the form $\sqrt{x^2 + y^2}$ means that it will be more decidedly diminished by reducing the larger of the two independent probable errors than by attempting to improve the more accurate measurement. Show that $\sqrt{5^2 + 2^2}$ is reduced by 41% if the 5 is changed to 2.5, but only by 5% if the 2 is changed to 1.

Draw a right-angled triangle with one side several (say 8 or 10) times as long as the other. Change the long side by making it a little longer or shorter and notice that the change in length of the hypothenuse is almost in exact proportion. Change the short side and notice that the hypothenuse is hardly affected at all.

The formula for the volume of a cylinder is $v = \pi l r^2$.

* These numbers refer, of course, to hundredths of a centimetre. It would be possible, but not advisable, to perform the operation $\sqrt{.04^2 + .03^2} = .05$ so as to obtain the answer in centimetres (188).

In determining this indirect measurement which of the two dimensions ought to be measured the more carefully? How much more carefully? Why?

234. Questions and Exercises.—1. Devise a method for determining velocities by direct measurement.

2. Each volume of a ten-volume encyclopedia has 5.0 cm. $\pm$ .1 cm. thickness of leaves between its two covers. Each cover has a thickness of .35 cm. $\pm$ .02 cm. How much more shelf-room than 57.00 cm. is it likely to need? What is the meaning of the word "likely" in the last sentence?

XXII. SYSTEMATIC AND CONSTANT ERRORS

Apparatus.—Clock, chronometer, or time circuit, giving audible seconds; watch with second hand; slide rule.

235. Definitions.—It has been shown that errors may be either accidental or constant. There is another class of errors, often included under the term constant errors, in which the error is not actually constant, nor does it vary according to the law of probability. This is the class of *systematic errors,* or errors that undergo a more or less regular change during the course of making a set of measurements. They may be subdivided into *progressive errors,* which show a steady increase (or decrease) from one determination to the next, and *periodic errors,* which increase for a number of measurements, then decrease, and then repeat the previous cycle or period.

236. A Test for Systematic Errors.—Where systematic errors are absent a comparison of any measurement of a series with the preceding one will tend to show an increase in the numerical value about as often as a decrease; a fact that can easily be tested by writing between each two successive values a plus sign, a minus sign, or a zero, according as the second value is respectively greater than, less than, or equal to, the first, and then comparing the number of the plus signs with that of the minus signs.

Where progressive errors have been greater than accidental errors there may be all plus signs or all minus signs as the result of applying the test. If the accidental errors are relatively large they will probably cause several of the signs to be plus and several minus, but the presence of a progressive error at the same time will cause one sign to appear more frequently than the other.

If the systematic errors are periodic there will be alternate groups of plus signs and minus signs, as is shown in the next table.

237. Example of a Systematic Error.—In an experiment in which water in a reservoir was drawn up into a tube by suction and successive readings of its height were made values having the following decimals were obtained in order: .76, .74, .70, .62, .63, .61, .55, .56, .51, .50, .44, .44, .39, .40, .35, .35. Are the results probably affected by progressive errors, or periodic errors, or neither? Use a graphic diagram if the question is hard to answer. What effect would you expect to result from a slight leakage of air into the upper part of the tube?

238. Example of a Periodic Error.—If the pivot of the second hand of a watch is not exactly in the center of the dial the indicated seconds will be subject to a periodic error. For example, if it is located too far to the right the hand may indicate 29 instead of 30 when it points downward, and 1 instead of 60 when it is pointing upward. That is, it will have a periodic error which will be a maximum (positive) at 60 seconds, a minimum (negative) at 30 seconds, and zero at 15 and 45 seconds (Fig. 64). From the illustration it is evident that the direction of displacement must be toward a point halfway between the positions at which the error is greatest and least.

Stand where you can hear the clock beat seconds and read the time indicated by your watch. Every seven seconds as indicated by the clock read the seconds and estimated tenths of a second from the watch and state the result to another student, who will take down the values in his notebook. After three or four minutes change places with him and note down the time as he reads it off. Every seven-second interval should have

its time by the watch noted, for a full period of seven minutes. It is advisable to practice the procedure beforehand until you are sure that you can estimate tenths of a second with reasonable accuracy. If your estimates

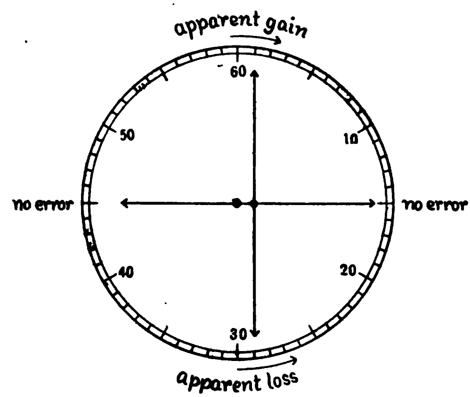


FIG. 64. ILLUSTRATION OF A PERIODIC ERROR.—An eccentric clock-hand will appear to be ahead of time, accurate, behind, accurate, ahead, etc., in the course of its rotation. A determination of the times at which the gain and the loss are most marked will enable the direction and amount of displacement to be found.

are predominantly 0.5 and 0.0 the results will not be satisfactory. In order to avoid losing count of the (audible) seconds while you are stating the time it may be advantageous to choose seven of your fingers and tap on the table with each in turn at one-second intervals. Concentrate your sense of sight on the watch and your sense of hearing on the beats of the clock.

See that you have the complete table of sixty values in your own notebook, and mark the observed tenths of a second with a plus sign where they increase from one observation to the next and with a minus sign where they decrease, as shown below. With most watches it

will be found that the second hand is not pivoted in the exact centre of the graduated circle and the periodic error will be shown very distinctly.

hr.min.sec.		*hr.min.sec.*		*hr.min.sec.*	
4:37:65.2	+	4:40:25.8	−	4:42:45.2	−
38:12.3	+	32.6	−	52.0	+
19.6	O	39.3	−	59.1	+
26.6	−	46.1	−	43:06.3	+
33.5	−	53.0	+	13.5	+
40.3	−	60.2	+	20.6	+
47.1	O	41:07.4	+	27.8	−
54.1	O	14.6	+	34.6	−
39:01.1	+	21.7	O	41.3	−
08.2	+	28.7	−	48.1	−
15.4	+	35.6	−	55.0	+
22.6	O	42.2	−	44:02.3	+
29.6	−	49.1	O	09.4	+
36.4	−	56.1	+	16.6	+
43.1	−	42:03.2	+	23.8	−
50.0	O	10.3	+	30.7	−
57.0	+	17.5	+	37.4	−
64.2	+	24.7	−	44.1	O
40:11.4	+	31.6	−	51.1	O
18.6	+	38.4	−	58.1	+
				4:45:05.2	

Apparent Time of Seven-Second Intervals.—The tabulated numbers are the times indicated by a watch at audible intervals that were known to be exactly seven seconds. The presence of a periodic error is shown by the tabular differences occurring in alternate groups of positive and negative values.

Draw a graphic diagram in which the abscissæ represent the integral part of the number of seconds in your table, and the ordinates represent the corresponding tenths of a second (Fig. 65). Draw a smooth curve to eliminate accidental errors in the determination of time.

Determine the direction and the amount of the displacement, and summarize the result by stating "the pivot of the second hand of the watch is displaced toward the figure ··· of the dial by an amount equal to the length of ··· seconds' divisions on the graduated circle."

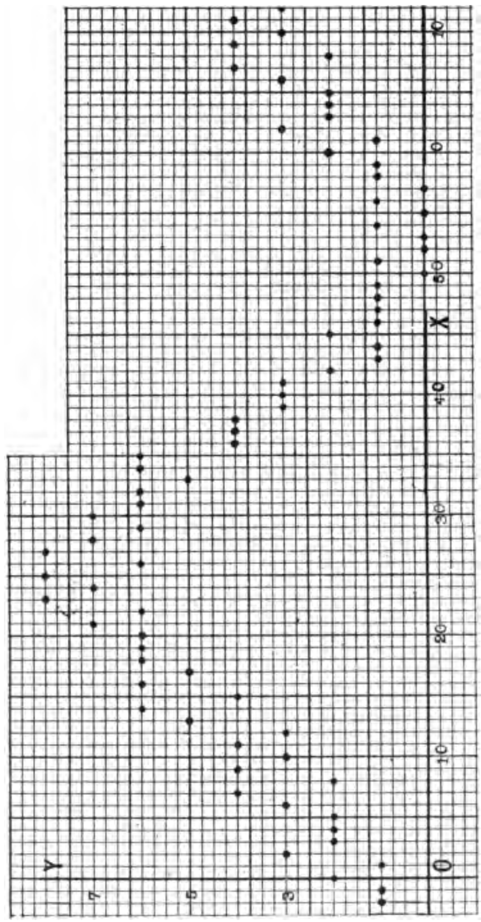


Fig. 65. Periodic Error of a Clock-Hand.—Each measurement of time that occurs in the previous table has its whole number of seconds represented by an x-value and its tenths of a second by a y-value.

Explain how the periodic error can be eliminated in case such a watch is used for determining intervals of time.

239. Example of a Progressive Error.—The list of figures given in § 237 was obtained from a determination of specific gravity by Hare's method. If the lower ends of two upright tubes dip into two separate reservoirs while their upper ends are both joined to a third tube from which the air can be partially exhausted it can easily be proved that the heights to which the fluids are raised will be inversely proportional to their densities; so that if a fluid whose density is unity is raised to a height h_1, and a heavier fluid to a lesser height h_2, the density or specific gravity of the latter must be h_1/h_2. The complete list of determinations of height included readings of both columns of liquid; they were made at approximately equal intervals of time, and in the order in which they are given in the table, *viz.*, 75.76, 73.06, 75.74, 73.04, 75.70, 73.00, 72.98, 72.95, 75.62, etc.

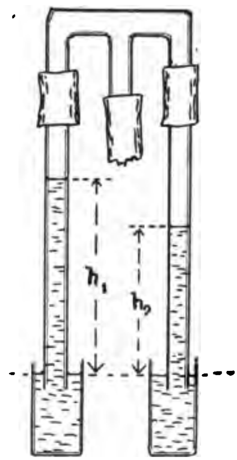


Fig. 66. Hare's Method of Balancing Columns.—The heights of the two liquids are inversely proportional to their densities.

If the density is calculated by dividing 75.76 by 73.06 it is evident that the progressive error will make the resulting figure too large, for the height of the water had fallen somewhat below 75.76 when the reading of the salt solution, 73.06, was taken; and if 75.74 is divided by 73.06 the progressive error will make the result too

small, for the salt solution did not stay at 73.06 while the reading 75.74 was being taken. Obviously the average of 75.76 and 75.74 must be divided by 73.06, or 75.74 must be divided by the average of 73.06 and 73.04, or some other combination used in which the *average* height of one column of liquid must have occurred at the same time as the average height of the other. This method of eliminating progressive errors is used in the process of weighing with a delicate balance and in many other processes of physical measurement.

What set of values near the end of the table can be used in the same way? Make five different calculations of density from successive parts of the table and see whether they show any evidence of *progressive* error.

240. Constant Errors.—It has already been stated that constant errors are more troublesome than accidental errors and that the latter give very little aid in determining the former. It is not the target (page 193) that is found from individual measurements but only the centre of clustering, and *characteristic deviations show only how close determinations come to one another, not how close they come to the truth.*

pure water	*salt sol tion*
75.76	
	73.06
75.74	
	73.04
75.70	
	73.00
	72.98
	.95
75.62	
	.94
.63	
.61	
	.89
	.88
.55	
.56	
	.84
	.84
.51	
.50	
	.79
	.78
.44	
.44	
	.75
	.70
.39	
.40	
	.77
	.77
.35	
.35	

Example of Heights in Hare's Method.

Some constant errors are easily corrected with the aid of theoretical considerations; others may be very difficult to eliminate. Unfortunately there is no infallible rule

for detecting them, and each experimental problem has its own special sources of error. The two beam-arms of a balance may be unequal, so that all weighings are proportionately erroneous; the end of a metre-stick may be worn, so that every setting of the zero-point is inaccurate; the neutral tint of litmus may be faultily judged, so that a chemical determination is biassed. Consider such a simple process as the determination of atmospheric pressure with a mercurial barometer. The vacuum at the top is never perfect and there is often capillary action, both making the reading too low. If the barometer and its attached scale do not hang vertically every apparent reading will be too high. The scale itself is too long or too short except at a single temperature, and the mercury may have its accepted standard density only at a different temperature from the one that it has when the observation is made. Even if its density is standard the height of a column that will give a definite pressure will depend upon the strength of gravitational attraction and this varies with the latitude and altitude of the instrument. If an aneroid barometer is to be used instead of a mercurial one its mechanism introduces still more sources of error.

It is evident that the amount of constant error will generally be varied by changing observers, apparatus, methods, and times of observation; and the more radically different the sets of conditions are made the better, in all probability, will be the mutual neutralization of constant errors when the weighted average is taken. In practice, the values for most of the constants of nature have been obtained under such varying conditions. Atomic weights are obtained from various interrelations of chemical compounds obtained from different sources and by different methods. The surface tension

of water may be measured by the hanging drop method, by the capillary wave method, by the vibrating jet method, etc. The size of the molecules of a gas may be calculated from the rate at which heat is conducted through them, from the *covolume constant, b,* of Van der Waal's equation, from experimental determinations of the viscosity of the gas, from measurements of the maximum density obtainable by cooling and liquefying or solidifying it, etc. If various determinations agree closely in spite of the employment of essentially different methods it becomes more probable that constant errors have been satisfactorily removed, but it can never be certain that all of these methods have not some common source of error which would be eliminated only by using some entirely different method. Constant watchfulness, as stated in § 162, and the exercise of good judgment are of the greatest importance in guarding against constant errors. If the student takes up further courses that involve accurate measurement he will usually find that various "sources of error" which have been found by previous experimenters will be explicitly stated. Many of them will be sources of constant error, and both his natural ability and his progress in learning will be put to the test in his management of them.

APPENDIX

TABLES

EXPLANATORY NOTES

Formulæ............................page 282

Equivalents............................page 283

The logarithm of each stated factor is given in another column for convenience of computation. The table of approximate equivalents is for use when no great accuracy is required.

Greek Alphabet........................page 284

Size of Errors..........................page 284

The words given in this table will help to fix the attention better than the numerical quantities.

Characteristic Deviations................page 284

General Sources of Error................ " 284

Density of Water....................... " 285

Notice that the density (mass per volume) of water (under atmospheric pressure) is never as great as unity.* A parallel column gives the specific gravity with reference to water at the temperature of maximum density.

Inverse Tangents and Circular Measure....page 285

Squares and Square Roots...........pages 286–287

The squares and square roots of all numbers are obtained with four-figure accuracy by using this table like any logarithm table. Complete five-figure and six-figure squares of all three-figure number are obtained

* The terms *density* and *specific gravity* are often confused, even in text books. Sometimes an arbitrary unit of volume is substituted for the cubic centimetre in order that the maximum density of water may appear to be unity, and a note is added to the effect that the density is stated in "grams per millilitre." (A *litre* is defined as the volume of 1000 grams of water at the temperature of maximum density.)

as follows: the last two figures of the required square (N^2) are printed in italic opposite the last two figures of N in the margin. These can then be used without ambiguity to correct the four-figure value obtained in the ordinary manner. *E. g.*, required the square of 417. The table gives the approximate value (pointed off by inspection) of 173900. Opposite 17 are found the italic figures *89*. The square is accordingly 173889; not 173989, for the latter would round off to 174000 instead of 173900.

The Probability Integral....................page 288

The tabular value gives the area under the curve $y = e^{-x^2}$ between the ordinates 0 and x in terms of the total area between $x = 0$ and $x = \infty$. In the small table of y and $x/.47694$ the value of l for Chauvenet's criterion will be found in the first column opposite the value of $1 - 1/2n$ in the second column.

Five-Place Logarithms....................page 289

This table is used like the four-place one. The first three figures of the natural number (antilogarithm) are found at the left, the fourth at the top, for the fifth interpolation is necessary.

If four-figure values are desired, round off a final five by increasing the previous figure unless the five is followed by a minus sign; *e. g.*, log 1.055 = .0233, but log 1.065 = .0273, not .0274.

Fifth Place of Logarithms.................page 289

Before annexing an *italicized* figure the four-figure value must be decreased by unity. *E. g.*, log 839 = 92376, not 92386; log 274 = .43775, but log 282 = 45025.

The "tabular differences" may be nearly as large as four hundred, but are easily handled *by logarithms* for purposes of interpolation.* *E. g.*, to find antilog 22222:

* See *Four-Place Logarithms*, page 279.

antilog 22011 = 166; antilog 22272 = 167; tab. dif. = 261; tabular excess of given logarithm = 211; 211/267 is about 0.8 (§ 14) and is found by subtracting log 267 (= 4265) from log 211 (= 3243), giving 8978 (= log 790); ∴ antilog 22222 = $166\frac{211}{267}$ = 166.79.

Exponentials . . . . page 289

A subscript number takes the place of a decimal point followed by the corresponding number of ciphers. *E. g.*, 0_44343 means 0.00004343, $_31234$ means .0001234, etc.

Notice that $e^{x+y} = e^x e^y$; for example, $e^{2.5}$ is 7.3891 × 1.6487, and $\log_{10}e^{2.5}$ is 1.0857.

Four-Place Logarithms . . . . pages 290–291

To find the logarithm of a given number: For the integral part ("characteristic") of the logarithm, count to the left from units' place to the first figure (other than zero) of the number, thus:

$$\overset{7}{9}\overset{6}{3}\overset{5}{0}\overset{4}{0}\overset{3}{0}\overset{2}{0}\overset{1}{0}\overset{0}{0}. \quad \text{or} \quad .\overset{0}{}\overset{-1}{0}\overset{-2}{0}\overset{-3}{0}\overset{-4}{3}05$$

log 93000000 = 7. + · · ·; log .000305 = − 4. + ... For the decimal part ("mantissa"), find the first (two) figures of the given number in the left-hand column (N) of the table and the third figure at the top of another column. The required mantissa will be found in line with the first two figures and in the column headed by the third. Consider the second or third figure, if lacking, to be zero. *E. g.*, log 7 = log 70 = log 700 = .8451; log .000023 = log .0000230 = − 5. + .3617. The mantissa is always kept positive, and a logarithm like the last is abbreviated $\bar{5}.3617$ to save space. If the number has four significant figures find the logarithms of the next smaller and next larger marginal numbers and assume that logarithmic differences are proportional to the corresponding numerical differences. Thus, 1.873 would be

located (on a scale) 3 tenths of the way from 1.87 to 1.88; therefore log 1.873 is likewise 3/10 of the way from 2718 to 2742, namely 2725. (Three tenths of the *tabular difference*, 24, will be found from the small marginal tables to be 7, and 2718 + 7 = 2725. The approximate tabular difference, D, is given for each line, so that only the final digits need be subtracted.) If the given number has 5 or more significant figures a table in which the logarithms are stated to 5 or more places must be used.

To find the number ("antilogarithm") that corresponds to a given logarithm: If the given logarithm does not occur in the body of the table determine its position in respect to the next higher and lower tabular logarithms and use proportional parts as before. *E. g.*, .1345 is found to be 10/32 of the way from .1335 to .1367, hence its antilogarithm will be $1.36\frac{10}{32}$, or 1.363. Notice that 10/32 can be reduced to tenths and 3/10 to twenty-fourths, mentally, by using the small multiplication tables (PP) in the margin.

Reciprocals are easily determined mentally by using a table of logarithms. *E. g.*, $1/e = 0.3678$. (Foot-note, page 281.)

The use of the small table of squares will be self-evident. Notice that the square of a number between 100 and 110, say of $100 + n$ or 107, consists of five figures which are, in order, 1, $2n$, n^2, or 1, 14, 49. The square of any number between 100 and 200 can be found by the same process, "carrying" mentally. Thus

$$
\begin{array}{rr@{\qquad\qquad}rr}
112^2 = & 1 & 173^2 = & 1 \\
 & 24 & & 146 \\
 & 144 & & 5329 \\
\hline
 & 12544 & & 29929
\end{array}
$$

If either $\Sigma(v^2)/n$ or $\Sigma(v^2)/n(n-1)$ is located between two consecutive numbers in the *third* column, $(n \pm 1/2)^2/(.67449)^2$, of the same table, then the value of $.67449\sqrt{\Sigma(v^2)/n}$ or $.67449\sqrt{\Sigma(v^2)/n(n-1)}$, as the case may be, will be found opposite it in the *first* column. A very rough mental calculation will prevent taking a value which is $\sqrt{10}$ times too large or small.

Constants .... page 292

The characteristics $\bar{1}$ and $\bar{2}$ have been replaced by 9 and 8 respectively.

Circular Functions .... page 292

In the table of circular functions the "radian value," natural sine, cosine, tangent, and cotangent are given for every degree of the quadrant (above 45° use the *lower* and *right-hand* margin), also the logarithmic sine and cosine. By subtracting the two latter from each other and from zero any of the six logarithmic functions may be obtained from the table by inspection.* Sines and

* When two logarithms are to be added or subtracted it will be found more convenient, after a little experience, to work from left to right than the reverse. This is especially easy in finding reciprocals by subtracting from zero (as in §68, no. 7): beginning at the left subtract each figure from 9, except the last one, which is to be subtracted from 10. For example,

$$\begin{aligned} \log 1 = 0 &= \bar{1}.\ 9\ 9\ 9\ 9\ 10 \\ \log \pi &= \underline{0.\ 4\ 9\ 7\ 1\ \ 5} \\ \therefore \log 1/\pi &= \bar{1}.\ 5\ 0\ 2\ 8\ \ 5 \end{aligned}$$

Try working the following exercises from left to right. Before each addition note whether the next pair of figures will add up to more than nine and so give "one to carry." If they add up to exactly nine, look a step farther to the right, and so on.

$$\begin{array}{r} 4\,1\,5\,6\,5\,2\,8\,4\,3\,6\,4\,6\,4\,4\,3\,6\,1\,4\,6 \\ \underline{\pm\ 3\,2\,2\,8\,3\,2\,8\,9\,4\,3\,2\,3\,8\,4\,2\,3\,8\,5\,6} \\ = \qquad\qquad\qquad\qquad\qquad\qquad \end{array}$$

In subtracting from left to right, before setting down each partial difference notice whether it will need to be decreased by unity on account of the figures that follow.

cosines of any intermediate values can safely be obtained by interpolation, and tangents up to tan 70°. For the sine, tangent, and numerical measure of a small angle the equations at the corners of the table should be used as factors. *E.g.*, sin 3′ = 3 × .000290888 = .000872664.

For inverse "radians" and tangents see page 285.

FORMULÆ

Thermometry

$$F = 9C/5 + 32$$
$$R = 4C/5$$
$$C = 5(F - 32)/9$$
$$C = 5R/4$$

Logarithms

$$\log_e x = \log_{10} x / \log_{10} e = 2.3025851 \log_{10} x$$
$$\log_{10} x = \log_e x / \log_e 10 = .4342945 \log_e x$$
$$\log_{10} 2.3025851 = .3622157$$
$$\log_{10} .4342945 = \bar{1}.6377843$$

Constants

$$\pi = 3.141593 = 180°$$
$$e = 2.718282$$
$$\pi^2 = 9.869604$$
$$\sqrt{\pi} = 1.772454$$
$$1/\pi = .3183099$$
$$\sqrt{2} = 1.414214$$
$$\sqrt{3} = 1.732051$$
$$\sqrt{5} = 2.236068$$
$$\sqrt{7} = 2.645751$$
$$\sqrt{10} = 3.162278$$

Mensuration

triangle: base, b; altitude, a; area, $ab/2$.

parallelogram: base, b; altitude, a; area, ab.

circle: radius, r; circumference, $2\pi r$; area, πr^2.

ellipse: major axis, $2a$; minor axis, $2b$; area, πab.

cylinder: radius, r; length, l; surface, $\pi r^2 + 2\pi rl + \pi r^2$; volume, $\pi r^2 l$.

cone: radius, r; height, h; surface, $\pi r^2 + \pi r\sqrt{r^2 + h^2}$; volume, $\pi r^2 h/3$.

pyramid: area of base, a; height, h; perimeter of base, p; slant height, s; surface, $ps/2$; volume, $ah/3$.

sphere: radius, r; surface, $4\pi r^2$; volume, $4\pi r^3/3$.

For δ-formulæ see page 88.
For American wire gauge see p. 101.

Equivalents

The best determination of the ratio of 1 metre to 1 inch is 39.37043, and this is the value generally adopted in scientific work. The legal relationship, however, is 1 metre = 39.37079 inches in Great Britain and 1 inch = 1/39.37000 metre in the United States, the "metre" being a number of standard inches (36ths of the Imperial Standard Yard) in the former case, and the "inch" being defined as a certain fraction of the standard metre (International Prototype) in the latter.

The accepted ratio of 1 pound to 1 kilogram is .4535924, and the derived equivalents given in the table have been calculated from these two ratios and the accepted relationship 1 litre = 1000.027 cm³.

The U. S. dry measures and the Imperial measures have been calculated from the assumptions that 1 U. S. bu. is equal to 2150.420 cu. in., and 1 Imperial gallon (namely, the volume of 10 av. lbs. of water at 62° F., barometer at 30 inches, weighed in air against brass weights) is equal to 4.545853 litres.

Unit	*Equivalent*	*Logarithm*
Centimetre	= 0.3937043 inch	5951701
	= 15531.64 λ (Cd [red] 15° 760 dry air)	1912173
Square cm	= 0.1550030 square inch	1903402
Cubic cm	= 0.999973 mL	9999883
	= 16.89407 impl. ♏	2277342
	= 16.23116 U. S. ♏	2103495
Drachm	= 3.887936 gm.	5897191
U. S. f. ʒ	= 3.696593 cm.³	5678017
Impl. f. ʒ	= 3.551543 cm.³	5504170
Foot	= 30.47973 cm.	4840111
Square ft	= 929.0138 cm.²	9680222
Cubic ft	= 28316.09 cm.³	4520333
Grain	= 64.79893 mgm.	8115678
Gram	= 15.43235 grains	1884322
Inch	= 2.539978 cm.	4048299
Square in	= 6.451487 cm.²	8096598
Cubic in	= 16.38663 cm.³	2144897
Kilogram	= 2.204622 av. lbs.	3433342
	= 2.679229 Troy lbs.	4280097
Kilometre	= 0.6213767 mi.	7933550
Litre (vol. of 1000 gm. H_2O).	= 1.056716 U. S. liquid qt.	0239583
	= 0.9082158 U. S. dry qt.	9581891
	= 0.8799239 impl. qt.	9444447
Metre	= 3.280869 ft.	5159889
	= 1.093623 yd.	0388676
Mile	= 1609.330 metres.	2066450
Millilitre (mL)	= 1.000027 cm.³	0000117
	= 16.23160 U. S. ♏	2103612
	= 16.89452 impl. ♏	2277459
Minim (♏) U. S.	= 0.06160990 cm.³	7896505
Impl.	= 0.05919238 cm.³	7722658
Ounce av.	= 28.34953 gm.	4525458
Troy	= 31.10348 gm.	4928090
U. S. f. ℥	= 29.57275 mL	4708917
Impl. f. ℥	1 av. oz. of water at 62° F. = 28.41234 cm.³	4535070
Pound av.	= 453.5924 gm.	6566658
Troy	= 373.2418 gm.	5719903
Quart U. S. dry	= 1101.192 cm.³	0418630
U. S. liq	= 946.3280 cm.³	9760417
Impl.	= 1.136599 cm.³	0556074
Ton long	= 1016.047 kgm.	0069138
Short	= 907.1848 kgm.	9576958
Metric (1000 kgm.)	= 2204.622 av. lb	3433342
Yard	= 91.439208 cm.	9611324

Approximate Equivalents

25 mm. = 1 inch	60 mgm. = 1 gr.	15 ♏ = 1 cm.³
10 cm. = 4 in.	15 gr. = 1 gm.	30 cm.³ = 1 fl. oz.
40 in. = 1 metre	30 gm. = 1 oz.	1 litre = 1 quart
8 km. = 5 miles	11 lb. = 5 kgm.	1000 cm.³ = 1 litre
1000 cm.² = 1 square ft.	15 lb./sq. in. = 1 atmo. = 1kgm./cm.²	

For slide-rule equivalents see page 97.

GREEK ALPHABET

Letter			Used as a symbol for
Α	α*	alpha	Rotation of polarised light; temperature coefficient of expansion; angle.
Β	β	beta	Coefficient of expansion; angle.
Γ	γ	gamma	Ratio of specific heats; angle.
Δ	δ	delta	A small quantity; a finite difference (Δ); difference of ...
Ε	ε	epsilon	2.7183.
Ζ	ζ	zeta	z (in co-ordinates).
Η	η	eta	Viscosity; efficiency-ratio; y (in co-ordinates).
Θ	θ	theta	Temperature; angle.
Ι	ι	iota	$\sqrt{-1}$; intensity of electric current.
Κ	κ	kappa	Electrical conductivity; magnetic susceptibility.
Λ	λ	lambda	Wave-length; latitude.
Μ	μ	mu	Index of refraction; coeff. of friction; .0001 cm. ($\mu\mu = 10^{-7}$ cm.); permeability.
Ν	ν	nu	Reluctivity ($= 1/\mu$).
Ξ	ξ	xi	x (in co-ordinates).
Ο	ο†	omicron	
Π	π	pi	3.1416; product of factors such as . . . (Π; *cf.* Σ).
Ρ	ρ	rho	Density; radius.
Σ	σ ς	sigma	Sum of terms such as ... (Σ); density of air (σ); Poisson's ratio (σ).
Τ	τ	tau	Time; temperature; torque.
Υ	υ	upsilon	Specific volume ($= 1/\rho$).
Φ	φ	phi	Angle; function of ...; flux.
Χ	χ	chi	Function of ....
Ψ	ψ	psi	Solid angle.
Ω	ω	omega	Resistance (Ω); angular velocity; dispersive power.

* Usually *written* with one pen-stroke, somewhat like α, to avoid confusion with italic *a*.
† Not used as a symbol, on account of liability to confusion with o and 0.

SIZE OF ERRORS

.03% to .1%	"*very small*"
.1% to .3%	"*small*"
.3% to 1%	"*moderate*"
1% to 3%	"*large*"
3% to 10%	"*very large*"

CHARACTERISTIC DEVIATIONS

$p/m^* = .4769363$	$\log p/m = 6784603$
$a/m = .5641895$	$\log a/m = 7514250$
$s/m = .7071066$	$\log s/m = 8494849$
$p/s = .6744898$	$\log p/s = 8289754$
$p/a = .8453475$	$\log p/a = 9270353$
$a/s = .7978846$	$\log a/s = 9019401$
$s/p = 1.482602$	$\log s/p = 1710246$
$a/p = 1.182944$	$\log a/p = 0729647$
$s/a = 1.253314$	$\log s/a = 0980599$

* m..."modulus", in the equation $y = e^{-x^2/m^2}$

GENERAL SOURCES OF ERROR

Linear scales not parallel; lack of verticality; etc.
Faulty standardization of standards of comparison.
Uneven or irregular subdivision of standards.
Inaccurate "coincidence" or "bisection."
Faulty estimation of tenths or other subdivisions.
Parallax in reading scales or the position of a pointer.
Assumed zero of a scale.
Friction (as in a balance) and "play" or "back-lash."

Density of Water

temp.	*density* (*gm. per cm.*^{3})	*spec. grav.* (:H_2O *at* 4° C.)
0°C	.999841	.99987
2	.999941	.99997
4	.999973	1.00000
6	.999941	.99997
8	.999849	.99988
10	.999700	.99973
12	.999498	.99952
14	.999244	.99927
16	.998943	.99897
18	.998595	.99862
20	.998203	.99823
22	.997770	.99780
24	.997296	.99732
26	.996783	.99681
28	.996232	.99626
30	.995646	.99567
32	.99502	.99505
34	.99437	.99440
36	.99768	.99371
38	.99296	.99299
40	.99221	.99224
42	.99144	.99147
44	.99063	.99066
46	.98979	.98982
48	.98893	.98896
50	.98804	.98807
52	.98712	.98715
54	.98618	.98621
56	.98522	.98525
58	.98422	.98425
60	.98321	.98324
62	.98217	.98220
64	.98110	.98113
66	.98002	.98005
68	.97891	.97894
70	.97778	.97781
72	.97663	.97666
74	.97545	.97548
76	.97426	.97429
78	.97304	.97307
80	.97180	.97183
82	.97054	.97057
84	.96927	.96930
86	.96797	.96800
88	.96665	.96668
90	.96531	.96534
92	.96396	.96399
94	.96258	.96261
96	.96119	.96122
98	.95978	.95981
100	.95835	.95838

Inverse Tangents and Circular Measure

n	*is the circular measure and tangent of*
.0001	0° 00′ 20″.6 = .000100
.0002	41 .3 = .000200
.0003	1 01 .9 = .000300
.0004	1 22 .5 = .000400
.0005	1 43 .1 = .000500
.0006	2 03 .8 = .000600
.0007	2 24 .4 = .000700
.0008	2 45 .0 = .000800
.0009	3 05 .6 = .000900
.001	0° 03′ 26″.3 = .001000
.002	6 52 .5 = .002000
.003	10 18 .8 = .003000
.004	13 45 .1 = .004000
.005	17 11 .3 = .005000
.006	20 37 .6 = .006000
.007	24 03 .9 = .007000
.008	27 30 .1 = .008000
.009	30 56 .4 = .009000

n	*is the circular measure of*	*is the tangent of*
.01	0° 34′ 22″.6	0° 34′ 22″.6
.02	1 08 45 .3	1 08 44 .7
.03	1 43 07 .9	1 43 06 .1
.04	2 17 30 .6	2 17 26 .2
.05	2 51 53 .2	2 51 44 .7
.06	3 26 15 .9	3 26 01 .1
.07	4 00 38 .5	4 00 15 .0
.08	4 35 01 .2	4 34 26 .1
.09	5 09 23 .8	5 08 34 .0
.10	5° 43′ 46″.5	5° 42′ 38″.1
.11	6 18 09 .1	6 16 38 .3
.12	6 52 31 .8	6 50 34 .0
.13	7 26 54 .4	7 24 24 .9
.14	8 01 17 1	7 58 10 .6
.15	8 35 39 .7	8 31 50 .8
.16	9 10 02 .4	9 05 25 .0
.17	9 44 25 .0	9 38 53 .0
.18	10 18 47 .7	10 12 14 .3
.19	10 53 10 .3	10 45 28 .7
.20	11° 27′ 33″.0	11° 18′ 35″.8
.21	12 01 55 .6	11 51 35 .2
.22	12 36 18 .2	12 24 26 .7
.23	13 10 40 .9	12 57 09 .9
.24	13 45 03 .5	13 29 44 .6
.25	14 19 26 .2	14 02 10 .5
.26	14 53 48 .8	14 34 27 .2
.27	15 28 11 .5	15 06 34 .5
.28	16 02 34 .1	15 38 32 .1
.29	16 36 56 .8	16 10 19 .8
.30	17° 11′ 19″.4	16° 41′ 57″.3
.31	17 45 42 .1	17 13 24 .4
.32	18 20 04 .7	17 44 40 .8
.33	18 54 27 .4	18 15 46 .4
.34	19 28 50 .0	18 46 40 .9
.35	20 03 12 .7	19 17 24 .2
.36	20 37 35 .3	19 47 56 .0
.37	21 11 58 .0	20 18 16 .1
.38	21 46 20 .6	20 48 24 .4
.39	22 20 43 .3	21 18 20 .8

n	*is the circular measure of*	*is the tangent of*
.40	22° 55′ 05″.9	21° 48′ 05″.1
.41	23 29 28 .6	22 17 37 .1
.42	24 03 51 .2	22 46 56 .7
.43	24 38 13 .9	23 16 03 .7
.44	25 12 36 .5	23 44 58 .2
.45	25 46 59 .2	24 13 39 .9
.46	26 21 21 .8	24 42 08 .7
.47	26 55 44 .5	25 10 24 .7
.48	27 30 07 .1	25 38 27 .6
.49	28 04 29 .7	26 06 17 .5
.50	28° 38′ 52″.4	26° 33′ 54″.2
.51	29 13 15 .1	27 01 17 .7
.52	29 47 37 .7	27 28 27 .9
.53	30 22 00 .4	27 55 24 .9
.54	30 56 23 .0	28 22 08 .6
.55	31 30 45 .6	28 48 38 .8
.56	32 05 08 .3	29 14 55 .8
.57	32 39 30 .9	29 40 59 .3
.58	33 13 53 .6	30 06 49 .4
.59	33 48 16 .2	30 32 26 .2
.60	34° 22′ 38″.9	30° 57′ 49″.5
.61	34 57 01 .5	31 22 59 .5
.62	35 31 24 .2	31 47 56 .1
.63	36 05 46 .8	32 12 39 .3
.64	36 40 09 .5	32 37 09 .3
.65	37 14 32 .1	33 01 25 .7
.66	37 48 54 .8	33 25 29 .3
.67	38 23 17 .4	33 49 19 .5
.68	38 57 40 .1	34 12 56 .5
.69	39 32 02 .7	34 36 20 .4
.70	40° 06′ 25″.4	34° 59′ 31″.3
.71	40 40 48 .0	35 22 29 .1
.72	41 15 10 .7	35 45 14 .0
.73	41 49 33 .3	36 07 46 .0
.74	42 23 56 .0	36 30 05 .2
.75	42 58 18 .6	36 52 11 .6
.76	43 32 41 .3	37 14 05 .4
.77	44 07 03 .9	37 35 46 .6
.78	44 41 26 .5	37 57 15 .2
.79	45 15 49 .2	38 18 31 .5
.80	45° 50′ 11″.8	38° 39′ 35″.3
.81	46 24 34 .5	39 00 26 .8
.82	46 58 57 .1	39 21 06 .3
.83	47 33 19 .8	39 41 10 .3
.84	48 07 47 .8	40 01 48 .9
.85	48 42 05 .1	40 21 52 .3
.86	49 16 27 .7	40 41 43 .9
.87	49 50 50 .4	41 01 23 .8
.88	50 25 13 .0	41 20 52 .0
.89	50 59 35 .7	41 40 08 .7
.90	51° 33′ 58″.3	41° 59′ 14″.0
.91	52 08 20 .9	42 18 07 .9
.92	52 42 43 .6	42 36 50 .6
.93	53 17 06 .3	42 55 22 .2
.94	53 51 28 .9	43 13 42 .7
.95	54 25 51 .6	43 31 52 .3
.96	55 00 14 .2	43 49 51 .1
.97	55 34 36 .9	44 07 39 .1
.98	56 08 59 .5	44 25 16 .6
.99	56 43 22 .2	44 42 43 .5

COMPLETE SQUARES UP TO 999² 4-FIG. SQUARES AND SQUARE ROOTS

N	0	1	2	3	4	5	6	7	8	9	D
0	00	01	04	09	16	25	36	49	64	81	
10	1000	1020	1040	1061	1082	1102	1124	1145	1166	1188	22
11	1210	1232	1254	1277	1300	1322	1346	1369	1392	1416	24
12	1440	1464	1488	1513	1538	1562	1588	1613	1638	1664	26
13	1690	1716	1742	1769	1796	1822	1850	1877	1904	1932	28
14	1960	1988	2016	2050	2074	2102	2132	2161	2190	2220	30
15	2250	2280	2310	2341	2372	2402	2434	2465	2496	2528	32
16	2560	2592	2624	2657	2690	2722	2756	2789	2822	2856	34
17	2890	2924	2958	2993	3028	3062	3098	3133	3168	3204	36
18	3240	3276	3312	3349	3386	3422	3460	3497	3533	3572	38
19	3610	3648	3686	3725	3764	3802	3842	3881	3920	3960	40
20	4000	4040	4080	4121	4162	4202	4244	4285	4326	4368	42
21	4410	4452	4494	4537	4580	4622	4666	4709	4752	4796	44
22	4840	4884	4928	4973	5018	5062	5108	5153	5198	5244	46
23	5290	5336	5382	5429	5476	5522	5570	5617	5664	5712	48
24	5760	5808	5856	5905	5954	6002	6052	6101	6150	6200	50
25	6250	6300	6350	6401	6452	6502	6554	6605	6656	6708	52
26	6760	6812	6864	6917	6970	7022	7076	7129	7182	7236	54
27	7290	7344	7398	7453	7508	7562	7618	7673	7728	7784	56
28	7840	7896	7952	8009	8066	8122	8180	8237	8294	8352	58
29	8410	8468	8526	8585	8644	8702	8762	8821	8880	8940	60
30	9000	9060	9120	9181	9242	9302	9364	9425	9486	9548	62
31	9610	9672	9734	9797	9860	9922	9986	1005	1011	1018	6 4
32	1024	1030	1037	1043	1050	1056	1063	1069	1076	1082	7
33	1089	1096	1102	1109	1116	1122	1129	1136	1142	1149	7
34	1156	1163	1170	1176	1183	1190	1197	1204	1211	1218	7
35	1225	1232	1239	1246	1253	1260	1267	1274	1282	1289	7
36	1296	1303	1310	1318	1325	1332	1340	1347	1354	1362	7
37	1369	1376	1384	1391	1399	1406	1414	1421	1429	1436	8
38	1444	1452	1459	1467	1475	1482	1490	1498	1505	1513	8
39	1521	1529	1537	1544	1552	1560	1568	1576	1584	1592	8
40	1600	1608	1616	1624	1632	1640	1648	1656	1665	1673	8
41	1681	1689	1697	1706	1714	1722	1731	1739	1747	1756	8
42	1764	1772	1781	1789	1798	1806	1815	1823	1832	1840	9
43	1849	1858	1866	1875	1884	1892	1901	1910	1918	1927	9
44	1936	1945	1954	1962	1971	1980	1989	1998	2007	2016	9
45	2025	2034	2043	2052	2061	2070	2079	2088	2098	2107	9
46	2116	2125	2134	2144	2153	2162	2172	2181	2190	2200	9
47	2209	2218	2228	2237	2247	2256	2266	2275	2285	2294	10
48	2304	2314	2323	2333	2343	2352	2362	2372	2381	2391	10
49	2401	2411	2421	2430	2440	2450	2460	2470	2480	2490	10
50	2500	2510	2520	2530	2540	2550	2560	2570	2581	2591	10
51	2601	2611	2621	2632	2642	2652	2663	2673	2683	2694	10
52	2704	2714	2725	2735	2746	2756	2767	2777	2788	2798	11
53	2809	2820	2830	2841	2852	2862	2873	2884	2894	2905	11
54	2916	2927	2938	2948	2959	2970	2981	2992	3003	3014	11

PP

	64	63	62	61	60
1	6.4	6.3	6.2	6.1	6.0
2	12.8	12.6	12.4	12.2	12.0
3	19.2	18.9	18.6	18.3	18.0
4	25.6	25.2	24.8	24.4	24.0
5	32.0	31.5	31.0	30.5	30.0
6	38.4	37.8	37.2	36.6	36.0
7	44.8	44.1	43.4	42.7	42.0
8	51.2	50.4	49.6	48.8	48.0
9	57.6	56.7	55.8	54.9	54.0

	59	58	57	56	55	54
1	5.9	5.8	5.7	5.6	5.5	5.4
2	11.8	11.6	11.4	11.2	11.0	10.8
3	17.7	17.4	17.1	16.8	16.5	16.2
4	23.6	23.2	22.8	22.4	22.0	21.6
5	29.5	29.0	28.5	28.0	27.5	27.0
6	35.4	34.8	34.2	33.6	33.0	32.4
7	41.3	40.6	39.9	39.2	38.5	37.8
8	47.2	46.4	45.6	44.8	44.0	43.2
9	53.1	52.2	51.3	50.4	49.5	48.6

	53	52	51	50	49	48
1	5.3	5.2	5.1	5.0	4.9	4.8
2	10.6	10.4	10.2	10.0	9.8	9.6
3	15.9	15.6	15.3	15.0	14.7	14.4
4	21.2	20.8	20.4	20.0	19.6	19.2
5	26.5	26.0	25.5	25.0	24.5	24.0
6	31.8	31.2	30.6	30.0	29.4	28.8
7	37.1	36.4	35.7	35.0	34.3	33.6
8	42.4	41.6	40.8	40.0	39.2	38.4
9	47.7	46.8	45.9	45.0	44.1	43.2

	47	46	45	44	43	42
1	4.7	4.6	4.5	4.4	4.3	4.2
2	9.4	9.2	9.0	8.8	8.6	8.4
3	14.1	13.8	13.5	13.2	12.9	12.6
4	18.8	18.4	18.0	17.6	17.2	16.8
5	23.5	23.0	22.5	22.0	21.5	21.0
6	28.2	27.6	27.0	26.4	25.8	25.2
7	32.9	32.2	31.5	30.8	30.1	29.4
8	37.6	36.8	36.0	35.2	34.4	33.6
9	42.3	41.4	40.5	39.6	38.7	37.8

	41	40	39	38	37	36
1	4.1	4.0	3.9	3.8	3.7	3.6
2	8.2	8.0	7.8	7.6	7.4	7.2
3	12.3	12.0	11.7	11.4	11.1	10.8
4	16.4	16.0	15.6	15.2	14.8	14.4
5	20.5	20.0	19.5	19.0	18.5	18.0
6	24.6	24.0	23.4	22.8	22.2	21.6
7	28.7	28.0	27.3	26.6	25.9	25.2
8	32.8	32.0	31.2	30.4	29.6	28.8
9	36.9	36.0	35.1	34.2	33.3	32.4

COMPLETE SQUARES UP TO 999²   4-FIG. SQUARES AND SQUARE ROOTS

N	0	1	2	3	4	5	6	7	8	9	*D*
55	30*25*	3036	3047	3058	3069	3080	3091	3102	3114	3125	11
56	31*36*	3147	3158	3170	3181	3192	3204	3215	3226	3238	11
57	32*49*	3260	3272	3283	3295	3306	3318	3329	3341	3352	12
58	33*64*	3376	3387	3399	3411	3422	3434	3446	3457	3469	12
59	34*81*	3493	3505	3516	3528	3540	3552	3564	3576	3588	12
60	36*00*	3612	3624	3636	3648	3660	3672	3684	3697	3709	12
61	37*21*	3733	3745	3758	3770	3782	3795	3807	3819	3832	12
62	38*44*	3856	3869	3881	3894	3906	3919	3931	3944	3956	13
63	39*69*	3982	3994	4007	4020	4032	4045	4058	4070	4083	13
64	40*96*	4109	4122	4134	4147	4160	4173	4186	4199	4212	13
65	42*25*	4238	4251	4264	4277	4290	4303	4316	4330	4343	13
66	43*56*	4369	4382	4396	4409	4422	4436	4449	4462	4476	13
67	44*89*	4502	4516	4529	4543	4556	4570	4583	4597	4610	14
68	46*24*	4638	4651	4665	4679	4692	4706	4720	4733	4747	14
69	47*61*	4775	4789	4802	4816	4830	4844	4858	4872	4886	14
70	49*00*	4914	4928	4942	4956	4970	4984	4998	5013	5027	14
71	50*41*	5055	5069	5084	5098	5112	5127	5141	5155	5170	14
72	51*84*	5198	5213	5227	5242	5256	5271	5285	5300	5314	15
73	53*29*	5344	5358	5373	5388	5402	5417	5432	5446	5461	15
74	54*76*	5491	5506	5520	5535	5550	5565	5580	5595	5610	15
75	56*25*	5640	5655	5670	5685	5700	5715	5730	5746	5761	15
76	57*76*	5791	5806	5822	5837	5852	5868	5883	5898	5914	15
77	59*29*	5944	5960	5975	5991	6006	6022	6037	6053	6068	16
78	60*84*	6100	6115	6131	6147	6162	6178	6194	6209	6225	16
79	62*41*	6257	6273	6288	6304	6320	6336	6352	6368	6384	16
80	64*00*	6416	6432	6448	6464	6480	6496	6512	6529	6545	16
81	65*61*	6577	6593	6610	6626	6642	6659	6675	6691	6708	16
82	67*24*	6740	6757	6773	6790	6806	6823	6839	6856	6872	17
83	68*89*	6906	6922	6939	6956	6972	6989	7006	7022	7039	17
84	70*56*	7073	7090	7106	7123	7140	7157	7174	7191	7208	17
85	72*25*	7242	7259	7276	7293	7310	7327	7344	7362	7379	17
86	73*96*	7413	7430	7448	7465	7582	7500	7517	7534	7552	17
87	75*69*	7586	7604	7621	7639	7656	7674	7691	7709	7726	18
88	77*44*	7762	7779	7797	7815	7832	7850	7868	7885	7903	18
89	79*21*	7939	7957	7974	7992	8010	8028	8046	8064	8082	18
90	81*00*	8118	8136	8154	8172	8190	8208	8226	8245	8263	18
91	82*81*	8299	8317	8336	8354	8372	8391	8409	8427	8446	18
92	84*64*	8482	8501	8519	8538	8556	8575	8593	8612	8630	19
93	86*49*	8668	8686	8705	8724	8742	8761	8780	8798	8817	19
94	88*36*	8855	8874	8892	8911	8930	8949	8968	8987	9006	19
95	90*25*	9044	9063	9082	9101	9120	9139	9158	9178	9197	19
96	92*16*	9235	9254	9274	9293	9312	9332	9351	9370	9390	19
97	94*09*	9428	9448	9467	9487	9506	9526	9545	9565	9584	20
98	96*04*	9624	9643	9663	9683	9702	9722	9742	9761	9781	20
99	98*01*	9821	9841	9860	9880	9900	9920	9940	9960	9980	20

PP

	35	34	33	32	31	30
1	3.5	3.4	3.3	3.2	3.1	3.0
2	7.0	6.8	6.6	6.4	6.2	6.0
3	10.5	10.2	9.9	9.6	9.3	9.0
4	14.0	13.6	13.2	12.8	12.4	12.0
5	17.5	17.0	16.5	16.0	15.5	15.0
6	21.0	20.4	19.8	19.2	18.6	18.0
7	24.5	23.8	23.1	22.4	21.7	21.0
8	28.0	27.2	26.4	25.6	24.8	24.0
9	31.5	30.6	29.7	28.8	27.9	27.0

	29	28	27	26	25	24
1	2.9	2.8	2.7	2.6	2.5	2.4
2	5.8	5.6	5.4	5.2	5.0	4.8
3	8.7	8.4	8.1	7.8	7.5	7.2
4	11.6	11.2	10.8	10.4	10.0	9.6
5	14.5	14.0	13.5	13.0	12.5	12.0
6	17.4	16.8	16.2	15.6	15.0	14.4
7	20.3	19.6	18.9	18.2	17.5	16.8
8	23.2	22.4	21.6	20.8	20.0	19.2
9	26.1	25.2	24.3	23.4	22.5	21.6

	23	22	21	20	19	18
1	2.3	2.2	2.1	2.0	1.9	1.8
2	4.6	4.4	4.2	4.0	3.8	3.6
3	6.9	6.6	6.3	6.0	5.7	5.4
4	9.2	8.8	8.4	8.0	7.6	7.2
5	11.5	11.0	10.5	10.0	9.5	9.0
6	13.8	13.2	12.6	12.0	11.4	10.8
7	16.1	15.4	14.7	14.0	13.3	12.6
8	18.4	17.6	16.8	16.0	15.2	14.4
9	20.7	19.8	18.9	18.0	17.1	16.2

	17	16	15	14	13	12
1	1.7	1.6	1.5	1.4	1.3	1.2
2	3.4	3.2	3.0	2.8	2.6	2.4
3	5.1	4.8	4.5	4.2	3.9	3.6
4	6.8	6.4	6.0	5.6	5.2	4.8
5	8.5	8.0	7.5	7.0	6.5	6.0
6	10.2	9.6	9.0	8.4	7.8	7.2
7	11.9	11.2	10.5	9.8	9.1	8.4
8	13.6	12.8	12.0	11.2	10.4	9.6
9	15.3	14.4	13.5	12.6	11.7	10.8

	11	10	9	8	7	6
1	1.1	1.0	0.9	0.8	0.7	0.6
2	2.2	2.0	1.8	1.6	1.4	1.2
3	3.3	3.0	2.7	2.4	2.1	1.8
4	4.4	4.0	3.6	3.2	2.8	2.4
5	5.5	5.0	4.5	4.0	3.5	3.0
6	6.6	6.0	5.4	4.8	4.2	3.6
7	7.7	7.0	6.3	5.6	4.9	4.2
8	8.8	8.0	7.2	6.4	5.6	4.8
9	9.9	9.0	8.1	7.2	6.3	5.4

THE PROBABILITY INTEGRAL $y = \frac{2}{\pi}\int_0^x e^{-x^2}dx$

The tabular value gives the area under the curve $y = e^{-x^2}$ between the ordinates 0 and x in terms of the total area between $x = 0$ and $x = +\infty$. For Chauvenet's criterion see p. 234.

x	0	1	2	3	4	5	6	7	8	9	D
0.0	00000	01128	02256	03384	04511	05637	06762	07886	09008	10128	1118
0.1	11246	12362	13476	14587	15695	16800	17901	18999	20094	21184	1086
0.2	22270	23352	24430	25502	26570	27633	28690	29742	30788	31828	1035
0.3	32863	33891	34913	35928	36936	37938	38933	39921	40901	41874	965
0.4	42839	43797	44747	45689	46623	47548	48466	49375	50275	51167	883
0.5	52050	52924	53790	54646	55494	56332	57162	57982	58792	59594	792
0.6	60386	61168	61941	62705	63459	64203	64938	65663	66378	67084	696
0.7	67780	68467	69143	69810	70468	71116	71754	72382	73001	73610	600
0.8	74210	74800	75381	75952	76514	77067	77610	78144	78669	79184	507
0.9	79691	80188	80677	81156	81627	82089	82542	82987	83423	83851	419
1.0	84270	84681	85084	85478	85865	86244	86614	86977	87333	87680	340
1.1	88020	88353	88679	88997	89308	89612	89910	90200	90484	90761	270
1.2	91031	91296	91553	91805	92051	92290	92524	92751	92973	93190	211
1.3	93401	93606	93807	94002	94191	94376	94556	94731	94902	95067	162
1.4	95229	95385	95538	95686	95830	95970	96105	96237	96365	96490	121
1.5	96611	96728	96841	96952	97059	97162	97263	97360	97455	97546	89
1.6	97635	97721	97804	97884	97962	98038	98110	98181	98249	98315	64
1.7	98379	98441	98500	98558	98613	98667	98719	98769	98817	98864	45
1.8	98909	98952	98994	99035	99074	99111	99147	99182	99216	99248	31
1.9	99279	99309	99338	99366	99392	99418	99443	99466	99489	99511	21
2.0*	99532	*55248	57195	59063	60858	62581	64235	65822	67344	68805	1400
2.1*	70205	71548	72836	74070	75253	76386	77472	78511	79505	80459	913
2.2*	81372	82244	83079	83878	84642	85373	86071	86739	87377	87986	582
2.3*	88568	89124	89655	90162	90646	91107	91548	91968	92369	92751	364
2.4*	93115	93462	93793	94108	94408	94694	94966	95226	95472	95707	223
2.5*	95930	96143	96345	96537	96720	96893	97058	97215	97364	97505	135
2.6*	97640	97767	97888	98003	98112	98215	98313	98406	98494	98578	79
2.7*	98657	98732	98802	98870	98933	98994	99051	99105	99156	99204	46
2.8*	99250	99293	99334	99372	99409	99443	99476	99507	99536	99563	26
2.9*	99589	99613	99636	99658	99679	99698	99716	99733	99750	99765	14
3.0*	99779	99793	99805	99817	99829	99839	99849	99859	99867	99876	8
3.1*	99884	99891	99898	99904	99910	99916	99921	99926	99931	99936	4
3.2*	99940	99944	99947	99951	99954	99957	99960	99962	99965	99967	2
3.3*	99969	99971	99973	99975	99977	99978	99980	99981	99982	99984	1
3.4*	99985	99986	99987	99988	99989	99989	99990	99991	99991	99992	1
3.5†	99993	†99309	99358	99403	99445	99485	99521	99555	99587	99617	27
3.6†	99644	99670	99694	99716	99736	99756	99773	99790	99805	99819	14
3.7†	99833	99845	99857	99867	99877	99886	99895	99903	99910	99917	6
3.8†	99923	99929	99934	99939	99944	99948	99952	99956	99959	99962	3
3.9†	99965	99968	99970	99973	99975	99977	99979	99980	99982	99983	2
4. †	99985	99993	99997	99999	00000	00000	00000	00000	00000	00000	
∞	1										

$\frac{x}{.47694}$	y
0.0	00000
0.5	26407
1.0	50000
1.1	54188
1.2	58171
1.3	61942
1.4	65498
1.5	68833
1.6	71949
1.7	74847
1.8	77528
1.9	79999
2.0	82266
2.1	84335
2.2	86216
2.3	87918
2.4	89450
2.5	90825
2.6	92051
2.7	93141
2.8	94105
2.9	94954
3.0	95698
3.1	96346
3.2	96910
3.3	97397
3.4	97817
3.5	98176
3.6	98482
3.7	98743
3.8	98962
3.9	99147
4.0*	*30228
4.1*	43137
4.2*	53857
4.3*	62718
4.4*	69998
4.5*	75957
4.6*	80816
4.7*	84759
4.8*	87935
4.9*	90500
5.0*	92549

* Beginning with $x = 2.01$ and $x/.47694 = 4.0$ the (seven-place) y-values have their first two figures (9's) omitted.

† Beginning with $x = 3.51$ the (nine-place) y-values have their first four figures (.9999) omitted.

Five-Place Logarithms

N	0	1	2	3	4	5	6	7	8	9	D
100	00000	00043	00087	00130	00173	00217	00260	00303	00346	00389	43
101	00432	00475	00518	00561	00604	00647	00689	00732	00775-	00817	43
102	00860	00903	00945	00988	01030	01072	01115-	01157	01199	01242	42
103	01284	01326	01368	01410	01452	01494	01536	01578	01620	01662	41
104	01703	01745	01787	01828	01870	01912	01953	01995-	02036	02078	41
105	02119	02160	02202	02243	02284	02325	02366	02407	02449	02490	41
106	02531	02572	02612	02653	02694	02735-	02776	02816	02857	02898	40
107	02938	02979	03019	03060	03100	03141	03181	03222	03262	03302	40
108	03342	03383	03423	03463	03503	03543	03583	03623	03663	03703	40
109	03743	03782	03822	03862	03902	03941	03981	04021	04060	04100	39

Exponentials

x	e^x	$\log_{10} e^x$	e^{-x}	e^{-x^2}
.0001	1.0001	$0_4 4343$	.9999	1.0000
.0002	1.0002	$0_4 8686$	.9998	1.0000
.0003	1.0003	$0_3 1303$	.9997	1.0000
.0004	1.0004	$0_3 1737$	.9996	1.0000
.0005	1.0005	$0_3 2171$	.9995	1.0000
.0006	1.0006	$0_3 2606$	.9994	1.0000
.0007	1.0007	$0_3 3040$	.9993	1.0000
.0008	1.0008	$0_3 3474$	.9992	1.0000
.0009	1.0009	$0_3 3909$	.9991	1.0000
.001	1.0010	$0_3 4343$	.9990	1.0000
.002	1.0020	$0_3 8686$	.9980	1.0000
.003	1.0030	$0_2 1303$	.9970	1.0000
.004	1.0040	$0_2 1737$	.9960	1.0000
.005	1.0050	$0_2 2171$	.9950	1.0000
.006	1.0060	$0_2 2606$	.9940	1.0000
.007	1.0070	$0_2 3040$	.9930	1.0000
.008	1.0080	$0_2 3474$	.9920	1.0000
.009	1.0090	$0_2 3909$	.9910	1.0000
.01	1.0100	$0_2 4343$	.9900	.9999
.02	1.0202	$0_2 8686$	.9802	.9996
.03	1.0305	$0_1 1303$	.9704	.9991
.04	1.0408	$0_1 1737$	.9608	.9984
.05	1.0513	$0_1 2171$	.9512	.9975
.06	1.0618	$0_1 2606$	.9418	.9964
.07	1.0725	$0_1 3040$	.9324	.9951
.08	1.0833	$0_1 3474$	.9231	.9936
.09	1.0942	$0_1 3909$	.9139	.9919
.1	1.1052	$0_1 4343$	.9048	.9900
.2	1.2214	$0_1 8686$	.8187	.9608
.3	1.3499	0.1303	.7408	.9139
.4	1.4918	0.1737	.6703	.8521
.5	1.6487	0.2171	.6065	.7788
.6	1.8221	0.2606	.5488	.6977
.7	2.0138	0.3040	.4966	.6126
.8	2.2255	0.3474	.4493	.5273
.9	2.4596	0.3909	.4066	.4449
1.	2.7183	0.4343	.3679	.3679
2.	7.3891	0.8686	.1353	$_1 1832$
3.	20.086	1.3029	.0498	$_3 1234$
4	54.598	1.7372	.0183	$_6 1125$
5.	148.41	2.1715	.0067	$_{10} 1389$
6.	403.43	2.6058	.0025	$_{15} 2320$
7.	1096.6	3.0401	.0009	$_{21} 5196$
8.	2981.0	3.4744	.0003	$_{27} 1604$
9.	8103.1	3.9087	.0001	$_{35} 8359$
10.	22029.	4.3429	.0000	$_{43} 7203$

A subscript 4 means .0000; etc.

Fifth Place of Logarithms

N	0	1234	5	6789
11	*9*	2280	0	*6985*
12	*8*	*9612*	1	*7019*
13	4	*7750*	3	4281
14	3	*2946*	*7*	*5269*
15	*6*	*8492*	3	2060
16	2	3294	*8*	1219
17	5	0355	4	1725
18	*7*	*8752*	*7*	1466
19	*5*	3060	3	*6775*
20	3	0503	*5*	*7765*
21	2	*8481*	4	*5664*
22	2	*9505*	*8*	1334
23	3	*1962*	*7*	1580
24	1	*2219*	*7*	4050
25	4	7023	4	4320
26	*7*	4060	5	*8135*
27	*6*	*7765*	3	1840
28	*6*	1592	4	*7890*
29	0	*9875*	2	*9627*
30	2	*7147*	0	*2456*
31	*6*	*6543*	1	*9639*
32	5	1605	*8*	2570
33	1	3445	4	4320
34	*8*	*5396*	2	*8383*
35	*7*	1470	3	*5789*
36	0	1110	*9*	*8753*
37	0	*7417*	3	*9494*
38	*8*	2603	*6*	9135
39	*6*	*8990*	0	*0987*
40	*6*	4318	*6*	*3962*
41	*8*	4050	5	*9481*
42	5	*8147*	*9*	1346
43	*7*	*8899*	*9*	*9876*
44	*5*	4208	6	3185
45	1	*8406*	1	*6271*
46	*6*	0482	*5*	*9257*
47	0	2468	*9*	1234
48	4	5555	4	4321
49	0	*8753*	1	8630
50	*7*	4073	*9*	*5162*
51	*7*	2726	1	5937
52	0	4703	*6*	*9136*
53	*8*	*9134*	*5*	*6789*
54	*9*	0000	0	*9987*
55	*6*	*5431*	*9*	7631
56	*9*	*6418*	5	2851
57	*7*	4051	*7*	*2838*
58	3	*8271*	*6*	0482
59	*5*	*9259*	2	5703
60	*5*	7024	*6*	*7902*
61	3	*4567*	*8*	*8999*
62	*9*	*9998*	*8*	*7765*
63	4	*3209*	*7*	*6420*
64	*8*	*6419*	*6*	3084
65	1	*8518*	4	*0739*
66	4	*0617*	2	*7383*
67	*7*	*2726*	0	*5937*
68	1	*5826*	*9*	*2692*
69	5	*8136*	*8*	1368
70	0	*2467*	*9*	0235
71	*6*	*7890*	1	1223
72	3	4444	4	4333
73	2	2100	*9*	*8764*
74	3	*2097*	*6*	4208
75	*6*	*4297*	5	2074
76	1	*8529*	*6*	3063
77	*9*	*5284*	0	*6284*
78	*9*	*5162*	*7*	*2738*
79	3	*8372*	*7*	1605
80	*9*	*3726*	0	4715
81	*9*	*2692*	*6*	*9258*
82	1	4703	*5*	*8135*
83	*8*	*2057*	*9*	*1346*
84	*8*	0134	*6*	*7801*
85	2	*3456*	*7*	*7899*
86	0	0111	2	2222
87	2	2211	1	*0099*
88	*8*	*8765*	4	3210
89	*9*	*8654*	2	*1986*
90	4	*2197*	5	*3196*
91	4	*2975*	2	0742
92	*9*	*6307*	4	1852
93	*8*	5285	1	*8407*
94	3	*9517*	3	*9517*
95	2	*8495*	0	*6172*
96	*7*	*2838*	3	*8382*
97	*7*	*2716*	0	*5948*
98	3	*7150*	4	*8260*
99	4	*7159*	2	*6037*
10	0	2043	*9*	1823

Before annexing an *italic* figure subtract 1 from the fourth decimal place.

Four-Place Logarithms

N	0	1	2	3	4	5	6	7	8	9	*D*
10	0000*	0043*	0086*	0128*	0170*	0212*	0253*	0294*	0334*	0374*	40
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	37
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	33
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	31
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	29
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	27
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	25
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	24
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	23
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	21
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	21
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	20
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	19
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	18
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	17
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	17
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	16
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	16
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	15
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	14
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	14
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	13
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	13
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	13
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	13
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	12
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	12
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	12
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	12
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	11
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	11
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	10
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	10
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	10
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	10
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	10
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	9
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	9
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	9
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	8
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	8
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	8
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	8

PP

	43	42	41	40
1	4.3	4.2	4.1	4.0
2	8.6	8.4	8.2	8.0
3	12.9	12.6	12.3	12.0
4	17.2	16.8	16.4	16.0
5	21.5	21.0	20.5	20.0
6	25.8	25.2	24.6	24.0
7	30.1	29.4	28.7	28.0
8	34.4	33.6	32.8	32.0
9	38.7	37.8	36.9	36.0

	39	38	37	36
1	3.9	3.8	3.7	3.6
2	7.8	7.6	7.4	7.2
3	11.7	11.4	11.1	10.8
4	15.6	15.2	14.8	14.4
5	19.5	19.0	18.5	18.0
6	23.4	22.8	22.2	21.6
7	27.3	26.6	25.9	25.2
8	31.2	30.4	29.6	28.8
9	35.1	34.2	33.3	32.4

	35	34	33	32
1	3.5	3.4	3.3	3.2
2	7.0	6.8	6.6	6.4
3	10.5	10.2	9.9	9.6
4	14.0	13.6	13.2	12.8
5	17.5	17.0	16.5	16.0
6	21.0	20.4	19.8	19.2
7	24.5	23.8	23.1	22.4
8	28.0	27.2	26.4	25.6
9	31.5	30.6	29.7	28.8

	31	30	29	28
1	3.1	3.0	2.9	2.8
2	6.2	6.0	5.8	5.6
3	9.3	9.0	8.7	8.4
4	12.4	12.0	11.6	11.2
5	15.5	15.0	14.5	14.0
6	18.6	18.0	17.4	16.8
7	21.7	21.0	20.3	19.6
8	24.8	24.0	23.2	22.4
9	27.9	27.0	26.1	25.2

	27	26	25	24
1	2.7	2.6	2.5	2.4
2	5.4	5.2	5.0	4.8
3	8.1	7.8	7.5	7.2
4	10.8	10.4	10.0	9.6
5	13.5	13.0	12.5	12.0
6	16.2	15.6	15.0	14.4
7	18.9	18.2	17.5	16.8
8	21.6	20.8	20.0	19.2
9	24.3	23.4	22.5	21.6

* Interpolated values are given on another page.

Four-Place Logarithms

N	0	1	2	3	4	5	6	7	8	9	*D*
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	8
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	7
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	7
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	6
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	6
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	5
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	
100	0000	0004	0009	0013	0017	0022	0026	0030	0035	0039	4

PP

	23	22	21	20
1	2.3	2.2	2.1	2.0
2	4.6	4.4	4.2	4.0
3	6.9	6.6	6.3	6.0
4	9.2	8.8	8.4	8.0
5	11.5	11.0	10.5	10.0
6	13.8	13.2	12.6	12.0
7	16.1	15.4	14.7	14.0
8	18.4	17.6	16.8	16.0
9	20.7	19.8	18.9	18.0

	19	18	17	16
1	1.9	1.8	1.7	1.6
2	3.8	3.6	3.4	3.2
3	5.7	5.4	5.1	4.8
4	7.6	7.2	6.8	6.4
5	9.5	9.0	8.5	8.0
6	11.4	10.8	10.2	9.6
7	13.3	12.6	11.9	11.2
8	15.2	14.4	13.6	12.8
9	17.1	16.2	15.3	14.4

	15	14	13	12
1	1.5	1.4	1.3	1.2
2	3.0	2.8	2.6	2.4
3	4.5	4.2	3.9	3.6
4	6.0	5.6	5.2	4.8
5	7.5	7.0	6.5	6.0
6	9.0	8.4	7.8	7.2
7	10.5	9.8	9.1	8.4
8	12.0	11.2	10.4	9.6
9	13.5	12.6	11.7	10.8

	11	10	9	8
1	1.1	1.0	0.9	0.8
2	2.2	2.0	1.8	1.6
3	3.3	3.0	2.7	2.4
4	4.4	4.0	3.6	3.2
5	5.5	5.0	4.5	4.0
6	6.6	6.0	5.4	4.8
7	7.7	7.0	6.3	5.6
8	8.8	8.0	7.2	6.4
9	9.9	9.0	8.1	7.2

	7	6	5	4
1	0.7	0.6	0.5	0.4
2	1.4	1.2	1.0	0.8
3	2.1	1.8	1.5	1.2
4	2.8	2.4	2.0	1.6
5	3.5	3 0	2.5	2.0
6	4.2	3.6	3.0	2.4
7	4.9	4.2	3.5	2.8
8	5.6	4.8	4.0	3.2
9	6.3	5.4	4.5	3.6

For natural logarithms see page 114.

Squares

n	n^2	$\frac{(n \pm \frac{1}{2})^2}{.67449^2}$	n	n^2	$\frac{(n \pm \frac{1}{2})^2}{.67449^2}$
		21762			77819
10	100	24234	60	3600	80456
11	121	29070	61	3721	83138
12	144	34345	62	3844	85864
13	169	40061	63	3969	88633
14	196	46215	64	4096	91447
15	225	52810	65	4225	94304
16	256	59843	66	4356	97206
17	289	67317	67	4489	10015
18	324	75230	68	4624	10314
19	361	83583	69	4761	10617
20	400	92375	70	4900	10925
21	441	10161	71	5041	11237
22	484	11128	72	5184	11554
23	529	12139	73	5329	11875
24	576	13194	74	5476	12200
25	625	14293	75	5625	12530
26	676	15436	76	5776	12864
27	729	16623	77	5929	13202
28	784	17854	78	6084	13545
29	841	19129	79	6241	13893
30	900	20448	80	6400	14244
31	961	21811	81	6561	14600
32	1024	23218	82	6724	14961
33	1089	24668	83	6889	15326
34	1156	26163	84	7056	15695
35	1225	27702	85	7225	16069
36	1296	29284	86	7396	16447
37	1369	30911	87	7569	16829
38	1444	32581	88	7744	17216
39	1521	34296	89	7921	17607
40	1600	36054	90	8100	18003
41	1681	37857	91	8281	18403
42	1764	39703	92	8464	18808
43	1849	41594	93	8649	19217
44	1936	43528	94	8836	19630
45	2025	45506	95	9025	20047
46	2116	47529	96	9216	20469
47	2209	49595	97	9409	20896
48	2304	51705	98	9604	21327
49	2401	53859	99	9801	21762
50	2500	56057	100	10000	22201
51	2601	58299	101	10201	22645
52	2704	60585	102	10404	23094
53	2809	62915	103	10609	23547
54	2916	65289	104	10816	24004
55	3025	67707	105	11025	24465
56	3136	70169	106	11236	24932
57	3249	72675	107	11449	25402
58	3364	75225	108	11664	25877
59	3481	77819	109	11881	26356

Constants

SYMBOL	CONSTANT	LOGARITHM
π	3.141593	0.4971499
$\angle 1$	$180°/\pi = 57°17'45''$	
	$=57°.29578$	1.7581226
	$=3437'.747$	3.5362739
	$=206264''.80625$	5.3144251
e	2.718282	0.4342945
M	0.4342945	9.6377843

Circular Functions

RAD	DEG	TAN	SIN	LOG SIN	LOG COS	COS	COT		$1' = 10^{-4} \times 2.90888$
0000	0	0000	0000	$-\infty$	0	1	∞	90	$\pi/2$
01745*	1	0175	0175	2419	9999	9998	5729	89	1553
03491	2	0349	0349	5428	9997	9994	2864	88	1536
05236	3	0524	0523	7188	9994	9986	1908	87	1518
06981	4	0699	0698	8436	9989	9976	1430	86	1501
08727	5	0875	0872	9403	9983	9962	1143	85	1484
1047	6	1051	1045	0192	9976	9945	9514	84	1466
1222	7	1228	1219	0859	9968	9925	8144	83	1449
1396	8	1405	1392	1436	9958	9903	7115	82	1431
1571	9	1584	1564	1943	9946	9877	6314	81	1414
1745	10	1763	1736	2397	9934	9848	5671	80	1396
1920	11	1944	1908	2806	9919	9816	5145	79	1379
2094	12	2126	2079	3179	9904	9781	4705	78	1361
2269	13	2309	2250	3521	9887	9744	4331	77	1344
2443	14	2493	2419	3837	9869	9703	4011	76	1326
2618	15	2679	2588	4130	9849	9659	3732	75	1309
2793	16	2867	2756	4403	9828	9613	3487	74	1292
2967	17	3057	2924	4659	9806	9563	3271	73	1274
3142	18	3249	3090	4900	9782	9511	3078	72	1257
3316	19	3443	3256	5126	9757	9455	2904	71	1239
3491	20	3640	3420	5341	9730	9397	2747	70	1222
3665	21	3839	3584	5543	9702	9336	2605	69	1204
3840	22	4040	3746	5736	9672	9272	2475	68	1187
4014	23	4245	3907	5919	9640	9205	2356	67	1169
4189	24	4452	4067	6093	9607	9135	2246	66	1152
4363	25	4663	4226	6259	9573	9063	2145	65	1134
4538	26	4877	4384	6418	9537	8988	2050	64	1117
4712	27	5095	4540	6570	9499	8910	1963	63	1100
4887	28	5317	4695	6716	9459	8829	1881	62	1082
5061	29	5543	4848	6856	9418	8746	1804	61	1065
5236	30	5774	5000	6990	9375	8660	1732	60	1047
5411	31	6009	5150	7118	9331	8572	1664	59	1030
5585	32	6249	5299	7242	9284	8480	1600	58	1012
5760	33	6494	5446	7361	9236	8387	1540	57	9948
5934	34	6745	5592	7476	9186	8290	1483	56	9774
6109	35	7002	5736	7586	9134	8192	1428	55	9599
6283	36	7265	5878	7692	9080	8090	1376	54	9425
6458	37	7536	6018	7795	9023	7986	1327	53	9250
6632	38	7813	6157	7893	8965	7880	1280	52	9076
6807	39	8098	6293	7989	8905	7771	1235	51	8901
6981	40	8391	6428	8081	8843	7660	1192	50	8727
7156	41	8693	6561	8169	8778	7547	1150	49	8552
7330	42	9004	6691	8255	8711	7431	1111	48	8378
7505	43	9325	6820	8338	8641	7314	1072	47	8203
7679	44	9657	6947	8418	8569	7193	1036	46	8029
7854	45	1	7071	8495	8495	7071	1000	45	7854
$1'' = 10^{-6} \times 4.84814$		COT	COS	LOG COS	LOG SIN	SIN	TAN	DEG	RAD

*log $\pi/180$ = log .01745329 = 8.2418774. For sines and tangents of numerical angles see p. 113.

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